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Homogeneous projective varieties with semi-continuous rank function

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Abstract. Let $\mathbb{X} \subset \mathbb{P}(V)$ be a projective variety, which is not contained in a hyperplane. Then every vector v in V may be written as a sum of vectors from the affine cone over $\mathbb{X}$. The minimal number of summands in such a sum is called the rank of v. In this paper, we classify all equivariantly embedded homogeneous projective varieties $\mathbb{X} \subset \mathbb{P}(V)$ whose rank function is lower semi-continuous. Classical examples are: the variety of rank one matrices (Segre variety with two factors) and the variety of rank one quadratic forms (quadratic Veronese variety). In the general setting, $\mathbb{X}$ is the orbit in $\mathbb{P}(V)$ of a highest weight line in an irreducible representation V of a reductive algebraic group G. Thus, our result is a list of all irreducible representations of reductive groups, for which the corresponding rank function is lower semi-continuous.

1. Introduction

Let V be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\mathbb{X} \subset \mathbb{P}(V)$ be a projective variety and $X \subset V$ be the affine cone over $\mathbb{X}$. Suppose that $\mathbb{X}$ is nondegenerate, i.e. it is not contained in a hyperplane. Then every vector $v \in V$ can be written as a linear combination of points of X and we have a well-defined function $\mathrm{rk} : \mathbb{P}(V) \to \mathbb{N}$, called *rank* (or $\mathbb{X}$-*rank*), given by

$$\mathrm{rk}[v] = \mathrm{rk}_{\mathbb{X}}[v] = \min\{r \in \mathbb{N} : v = x_1 + \ldots + x_r,\ \text{with } x_j \in X\},$$

where $[v] \in \mathbb{P}(V)$ denotes the projective point corresponding to a non-zero vector v. The *rank sets* in $\mathbb{P}(V)$ with respect to $\mathbb{X}$ are defined as

$$\mathbb{X}_r = \{[v] \in \mathbb{P}(V) : \mathrm{rk}[v] = r\}.$$

The *secant varieties* of $\mathbb{X}$ are defined as the Zariski closure

$$\sigma_r(\mathbb{X}) = \overline{\bigcup_{s \le r} \mathbb{X}_s}.$$

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The *border rank* of $[v] \in \mathbb{P}(V)$ is defined as

$$\underline{\mathrm{rk}}[v] = \underline{\mathrm{rk}}_{\mathbb{X}}[v] = \min\{r \in \mathbb{N} : [v] \in \sigma_r(\mathbb{X})\}.$$

Let $\mathrm{Aut}(\mathbb{X}) \subset SL(V)$ denote the group of linear automorphisms of $\mathbb{X}$. Then rank and border rank are $\mathrm{Aut}(\mathbb{X})$-invariant, and hence $\mathrm{Aut}(\mathbb{X})$ acts on $\mathbb{X}_r$ and $\sigma_r(\mathbb{X})$.

Definition 1.1. We call $\mathbb{X}$ *rs-continuous* if $\mathrm{rk}_{\mathbb{X}}$ is a lower semi-continuous function on $\mathbb{P}(V)$, i.e. if rank and border rank defined by X coincide. Otherwise, we say that $\mathbb{X}$ is *r-discontinuous*. We call $[v] \in \mathbb{P}(V)$ *exceptional* if $\mathrm{rk}_{\mathbb{X}}[v] \neq \underline{\mathrm{rk}}_{\mathbb{X}}[v]$.

Our goal is to classify all rs-continuous varieties belonging to a certain class, namely, the homogeneous rs-continuous varieties. By a homogeneous projective variety we mean an equivariantly embedded variety $\mathbb{X} \subset \mathbb{P}(V)$ whose automorphism group is transitive. In such a case the (linear) automorphism group G is always semisimple and $\mathbb{X} \cong G/P$, where P is a parabolic subgroup of G. In other words, $\mathbb{X}$ is a flag variety of G. Furthermore, when the embedding is nondegenerate, V is an irreducible G-module, and so it is determined up to isomorphism by its highest weight, say λ, so that $V \cong V(\lambda)$. Then $\mathbb{X}$ can be viewed as the orbit of a highest weight line $\mathbb{X} = G[v^{\lambda}] \subset \mathbb{P}(V)$. Conversely, if G is a semisimple algebraic group and $V = V(\lambda)$ is an irreducible G-module, then $\mathbb{P}(V)$ contains a unique closed G-orbit~—~the orbit of a highest weight line; we denote this orbit by $\mathbb{X}(G, V)$. It is not always true that G is the full automorphism group of $\mathbb{X}(G, V)$. For instance, the symplectic group Sp_{2n} acts transitively on the projective space $\mathbb{P}(\mathbb{F}^{2n})$, but the full linear automorphism group is SL_{2n}. Our approach is to classify all irreducible representations (G, V) such that $\mathbb{X}(G, V)$ is rs-continuous. Then the list of rs-continuous homogeneous projective varieties is obtained by dropping the redundancies. For brevity of expression, we shall call a representation (G, V) *rs-continuous*, if the corresponding variety $\mathbb{X}(G, V)$ is rs-continuous.

Before stating our classification theorem, let us mention some classical examples, where the terminology stems from.

Let $V = \mathbb{F}^m \otimes \mathbb{F}^n$ be the space of $m \times n$-matrices. On V we have the classical notion of rank of a matrix. Let $\mathbb{X} \subset \mathbb{P}(V)$ denote the variety of matrices of rank 1; $\mathbb{X}$ can be defined by the vanishing of all 2×2-minors and is also known as the Segre variety of simple tensors $\mathbb{X} = \mathrm{Segre}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$. It is well-known that every matrix of rank r can be written as a sum of r matrices of rank 1. The set of matrices of rank r or less is the common zero-locus of all $(r+1) \times (r+1)$-minors and hence is a closed set, which equals the secant variety $\sigma_r(\mathbb{X})$. Hence $\mathbb{X}$ is rs-continuous. The automorphism group of $\mathbb{X}$ equals $PSL_m \times PSL_n$ in this case.

A similar situation occurs for symmetric and for skew symmetric matrices. The corresponding varieties are the quadratic Veronese embedding $\mathrm{Ver}_2(\mathbb{P}^{n-1}) \subset \mathbb{P}(S^2\mathbb{F}^n)$ and the Grassmannian of planes $\mathrm{Gr}_2(\mathbb{F}^n) \subset \mathbb{P}(\Lambda^2\mathbb{F}^n)$. In both cases we have a linear action of SL_n preserving $\mathbb{X}$ and yielding the full automorphism group of $\mathbb{X}$. Both varieties are rs-continuous.

Perhaps the simplest r-discontinuous homogeneous variety is the twisted cubic curve: $\mathbb{X} = \mathrm{Ver}_3(\mathbb{P}^1) \subset \mathbb{P}(S^3\mathbb{F}^2)$. The respective automorphism group $G = PSL_2$ has three orbits in $\mathbb{P}(S^3\mathbb{F}^2)$, which can be written as $G[x^3]$, $G[x^2y]$ and $G[x^3 + y^3]$, where x, y is an arbitrary basis of $\mathbb{F}^2$. Indeed, an element of $\mathbb{P}(S^3\mathbb{F}^2)$ can be written as a product $[l_1l_2l_3]$, with $[l_j] \in \mathbb{P}^1$, and there are three possibilities: either $[l_1] = [l_2] = [l_3]$, or $[l_1] = [l_2] \neq [l_3]$, or all three are distinct. Since PSL_2 acts transitively on triples of distinct points on $\mathbb{P}^1$, we have three orbits in $\mathbb{P}(S^3\mathbb{F}^2)$. It is easy to see that the ranks are 1, 3 and 2, respectively. However, we have $\sigma_2(\mathbb{X}) = \overline{G[x^3 + y^3]} = \mathbb{P}(S^3\mathbb{F}^3)$ and hence $[x^2y]$ is exceptional and $\mathbb{X}$ is r-discontinuous.

Let us notice that, in the above example, $\overline{G[x^2y]}$ is the tangential variety of $\mathbb{X}$, which is clearly contained in the secant variety $\sigma_2(\mathbb{X})$. In fact it is a general phenomenon, that exceptional points do appear in the tangential variety, whenever they exist. This fact is in the basis of our methods.

Now, we formulate our main result.

Theorem 1.1. *Let G be a semisimple algebraic group and V be a finite dimensional irreducible G-module, with* $\dim V \geq 2$. *Then the closed G-orbit $\mathbb{X}(G, V) \subset \mathbb{P}(V)$ is rs-continuous if and only if the pair (G, V) appears in the following table.*

Group G	*Representation V*	*Highest weight of V*
	Simple classical groups	
SL_n	$\mathbb{F}^n$, $(\mathbb{F}^n)^*$, $(\Lambda^2\mathbb{F}^n)$, $(\Lambda^2\mathbb{F}^n)^*$, $S^2\mathbb{F}^n$, $(S^2\mathbb{F}^n)^*$, $\mathfrak{sl}_n$	π_1, π_{n-1}, π_2, π_{n-2}, $2\pi_1$, $2\pi_{n-1}$, $\pi_1 + \pi_{n-1}$
SO_n	$\mathbb{F}^n$, $RSpin_n (n \leq 10)$	π_1, $\pi_{\frac{n}{2}} (2 \mid n)$, $\pi_{\frac{n}{2}-1} (2 \mid n)$, $\pi_{\frac{n-1}{2}} (2 \nmid n)$
Sp_{2n}	$\mathbb{F}^{2n}$, $\Lambda^2_0\mathbb{F}^{2n}$, $S^2\mathbb{F}^{2n} \cong \mathfrak{sp}_{2n}$	$\pi_1, \pi_2, 2\pi_1$
	Simple exceptional groups	
E_6	$\mathbb{F}^{27}$, $(\mathbb{F}^{27})^*$	π_1, π_5
F_4	$\mathbb{F}^{26}$	π_1
G_2	$\mathbb{F}^7$	π_1
	Non-simple groups	
$SL_m \times SL_n$	$\mathbb{F}^m \otimes \mathbb{F}^n$	$\pi_1 \oplus \pi_1$
$SL_m \times Sp_{2n}$	$\mathbb{F}^m \otimes \mathbb{F}^{2n}$	$\pi_1 \oplus \pi_1$
$Sp_{2m} \times Sp_{2n}$	$\mathbb{F}^{2m} \otimes \mathbb{F}^{2n}$	$\pi_1 \oplus \pi_1$

, (1)

where by $RSpin_n$ we denote (any) spinor representation of the simply connected cover of SO_n, by $\Lambda^2_0\mathbb{F}^{2n}$ we denote the second fundamental representation of Sp_{2n} (which is identified with a hyperplane in $\Lambda^2\mathbb{F}^{2n}$), by $\mathbb{F}^{27}$ we denote any one of the two smallest fundamental representations of E_6, by $\mathbb{F}^{26}$ we denote the smallest fundamental representation of F_4, by $\mathbb{F}^7$ we denote the smallest fundamental representation of G_2. In the third column we list the highest weight of V with respect to G' (we use here and throughout the paper the numbering convention of [27, p. 294]).

Moreover, if (G, V) is r-discontinuous, then it contains an exceptional vector of border rank 2.

We give a proof of Theorem 1.1 in Sect. 3. As a corollary, we obtain the list of homogeneous projective varieties given below. The list of varieties is shorter, because, as mentioned above, in certain cases there are subgroups of the automorphism group of the variety acting transitively.

Corollary 1.2. *The rs-continuous projective varieties* $\mathbb{X} \subset \mathbb{P}(V)$ *with transitive linear automorphism group are the following:*

Notation for $\mathbb{X}$	Ambient $\mathbb{P}(V)$	Group G	Max $\mathrm{rk}_{\mathbb{X}}$
$\mathbb{P}(\mathbb{F}^n)$	$\mathbb{P}(\mathbb{F}^n)$	SL_n	1
$\mathrm{Ver}_2(\mathbb{P}(\mathbb{F}^n))$	$\mathbb{P}(S^2\mathbb{F}^n)$	SL_n	n
$\mathrm{Gr}_2(\mathbb{F}^n)$	$\mathbb{P}(\Lambda^2\mathbb{F}^n)$	SL_n	$\lfloor \frac{n}{2} \rfloor$
$\mathrm{Fl}(1, n-1; \mathbb{F}^n)$	$\mathbb{P}(\mathfrak{sl}_n)$	SL_n	n
Q^{n-2}	$\mathbb{P}(\mathbb{F}^n)$	SO_n	2
S^{10}	$\mathbb{P}(\mathbb{F}^{16})$	$Spin_{10}$	2
$\mathrm{Gr}_\omega(2, \mathbb{F}^{2n})$	$\mathbb{P}(\Lambda_0^2\mathbb{F}^{2n})$	Sp_{2n}	n
E^{16}	$\mathbb{P}(\mathbb{F}^{27})$	E_6	3
F^{15}	$\mathbb{P}(\mathbb{F}^{26})$	F_4	3
$\mathrm{Segre}(\mathbb{P}(\mathbb{F}^m) \times \mathbb{P}(\mathbb{F}^n))$	$\mathbb{P}(\mathbb{F}^m \otimes \mathbb{F}^n)$	$SL_m \times SL_n$	$\min\{m, n\}$

It is natural to ask whether rs-continuous varieties admit another general characterization. In fact, the starting point of our study was a result by Buczyński and Landsberg [4], based on previous work by Landsberg and Manivel [24]. This result exhibits a remarkable class of rs-continuous homogeneous varieties∼—∼the subcominuscule varieties. Recall that a variety $\mathbb{X} \subset \mathbb{P}(V)$ is called subcominuscule if it is the variety associated to the isotropy representation of an irreducible Hermitian symmetric space S, i.e. $\mathbb{X} = \mathbb{X}(G, V)$, with G being the complexification of the semisimple part of the isotropy subgroup the isometry group of S, and V being the tangent space. Then [4, Prop. 4.1] states that $\mathrm{rk}_{\mathbb{X}} = \underline{\mathrm{rk}}_{\mathbb{X}}$ and, furthermore, the G-orbits in $\mathbb{P}(V)$ are exactly the rank sets $\mathbb{X}_r$. It is then natural to ask: are there other rs-continuous homogeneous varieties besides the subcominuscule ones? There are. It was shown by Kaji and Yasukura [20], that the adjoint variety $\mathbb{X}(G, \mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ is rs-continuous if and only if $\mathfrak{g}$ is of type A_n or C_n; see also [3,19]. The adjoint variety of type C_n is just the quadratic Veronese variety, which is subcominuscule with respect to its automorphism group. However, the adjoint variety of type A_n is not subcominuscule. The non-subcominuscule varieties in our list are $\mathrm{Fl}(1, n-1; \mathbb{F}^n)$, $\mathrm{Gr}_\omega(2, \mathbb{F}^{2n})$ and F^{15}. All these exceptions are, however, hyperplane sections in subcominuscule varieties. Conversely, the homogeneous hyperplane section of subminuscule varieties are the above three and the quadric Q^{n-1} viewed as a hyperplane section in $\mathrm{Ver}_2(\mathbb{P}^n)$. This follows from a classification of homogeneous hyperplane sections in homogeneous projective varieties given by Watanabe [31]. Thus our theorem implies that, if $\mathbb{X}$ is a homogeneous hyperplane section in a subcominuscule variety $\tilde{\mathbb{X}}$ in its minimal projective embedding, then $\mathbb{X}$ is rs-continuous. So we can formulate the following:

Corollary 1.3. *A homogeneous projective variety* $\mathbb{X} \subset \mathbb{P}(V)$ *is rs-continuous if and only if it is either a subcominuscule variety, or a hyperplane section in a subcominuscule variety* $\tilde{\mathbb{X}} \subset \mathbb{P}(\tilde{V})$ *in its minimal projective embedding. In the latter case, the rank function of* $\mathbb{X}$ *is equal to the restriction of the rank function of*

$\tilde{\mathbb{X}}$ and the secant varieties of $\mathbb{X}$ are equal to the intersections of the secant varieties of $\tilde{\mathbb{X}}$ with $\mathbb{P}(V)$.

Since our approach is to start with a representation (G, V) rather than with a homogeneous variety $\mathbb{X}$, we have found the following observations useful. If (G, V) is a representation such that G acts spherically on the projective space $\mathbb{P}(V)$, then the variety $\mathbb{X}(G, V)$ is subcominuscule, i.e. $(Aut(\mathbb{X}), V)$ is a subminuscule representation. This follows directly from the classification of spherical representations given by Kac [18], see also [21].

Remark 1.1. Since spherical representations have finitely many orbits and, by the above corollary, rs-continuous representations are not far away from spherical, it makes sense to ask whether this class of representations is related to the well-known class of projective representations with finitely many orbits. The fact that twisted cubic is not rs-continuous together with the fact that $PSL_n(n \geq 3)$ has infinitely many orbits on $\mathbb{P}(\mathfrak{sl}_n)$ show that the class of rs-continuous representations neither contains, nor is contained in, the class of projective representations with finitely many orbits. Let us notice, however, that the automorphism group of an rs-continuous homogeneous projective variety has finitely many orbits in the projective space if and only if the variety is subcominuscule.

One of the most important questions, which is asked in the studies of secant varieties and rank is: what are the ideals of secant varieties? It is known, by a result of Kostant, that the ideal of $\mathbb{X}$ is generated in degree 2, by the appropriate generalization of the Plücker equations; cf. [23, Th. 16.2.2.6]. It was shown by Landsberg and Manivel [24], that, for a subcominuscule variety $\mathbb{X} \subset \mathbb{P}(V)$, the ideal of the r-th secant variety $\sigma_r(\mathbb{X})$ is generated in degree $r+1$ by the $(r-1)$-th prolongation of the generating set of the ideal of $\mathbb{X}$, which is defined as $I_2(\mathbb{X})^{(r-1)} = (I_2(\mathbb{X}) \otimes S^{r-1}V^*) \cap S^{r+1}V^*$. Our theorem has the following

Corollary 1.4. *If $\mathbb{X} \subset \mathbb{P}(V)$ is an rs-continuous homogeneous variety, then the ideal of $\sigma_r(\mathbb{X})$ is generated in degree $r+1$ by the prolongation $I_2(\mathbb{X})^{(r-1)}$.*

Let us emphasize, that the second secant variety $\sigma_2(\mathbb{X})$ plays a prominent role both in this paper and in the literature. Sometimes the second secant variety is called just "the secant variety". This variety is much more accessible than the higher secant varieties: for example for a simple G-module the second secant variety has an open G-orbit [32, Ch. III, Thm 1.4]. Furthermore, we have $\sigma_2(\mathbb{X}) = \mathbb{X}_2 \cup T\mathbb{X}$, where $T\mathbb{X}$ is the tangential variety of $\mathbb{X}$, cf. [23, 8.1]. It turns out that, if exceptional points exist, they are always present in $T\mathbb{X}$.

For a systematic treatment, as well as an extensive bibliography, on secant varieties and rank we refer the reader to the recent book of Landsberg [23]. The general theory of secant varieties allows one to deduce rs-continuity for varieties of small codimension, see Corollary 2.3 here. However, this criterion is applicable to relatively few homogeneous varieties, and to none of the more difficult cases.

The paper is organized as follows. In Sect. 2.1, we recall the notions of secant varieties, rank and border rank, with their basic properties. In Sect. 2.2, we recall some basic notions about algebraic groups: Borel subgroup, Cartan subgroup, weight lattice, root system, Weyl chamber. We also introduce the notion of chopping (this is a simple combinatorial procedure) and provide some facts on $\mathbb{X}_2(G, V)$ and $\sigma_2(\mathbb{X}(G, V))$ playing a crucial role in this paper.

In Sect. 3, we present a plan of our proof of Theorem 1.1, the main theorem of our article. Essentially, this proof is a compilation of Propositions 4.6, 4.8 and Theorems 6.1 and 7.1. We prove Propositions 4.6, 4.8 in Sect. 4. We prove Theorems 6.1, 7.1 in Sects. 6 and 7 respectively.

In Sect. 5, we prove a strong necessary condition for rs-continuity of a representation in terms of its choppings. This is formulated in Proposition 5.1.

The main statement of Sect. 6 is Theorem 6.1. In this theorem we find out which fundamental representations of classical groups are rs-continuous and which are r-discontinuous. This is done in the following way: for the fundamental modules

$$\mathbb{F}^n,\ \Lambda^2\mathbb{F}^n \text{ for } SL_n,\ \Lambda^3\mathbb{F}^6 \text{ for } SL_6,\ \Lambda^2_0\mathbb{F}^{2n} \text{ and } \Lambda^3_0\mathbb{F}^{2n} \text{ for } Sp_{2n},$$
$$\Lambda^2\mathbb{F}^n \text{ for } SO_n,\ RSpin_n \text{ for } Spin_n (n \leq 12)$$

we check rs-continuity/r-discontinuity in a straightforward way. From these data we deduce r-discontinuity of all other modules using the notion of chopping.

The main statement of Sect. 7 is Theorem 7.1. In this theorem we find out which fundamental representations of exceptional groups are rs-continuous and which are r-discontinuous. This is done via case by case checking of the 27 fundamental representations of the five exceptional Lie algebras. For any such representation we find some arguments by which it is r-discontinuous/rs-continuous. For most of the representations the arguments are quite short, using chopping or a reference, but for three representations:

$$V(\pi_1), V(\pi_2) \text{ for } F_4 \text{ and } V(\pi_1) \text{ for } E_7$$

we are able to find only relatively long arguments presented in the corresponding subsections.

2. Preliminaries

In this section, we recall some definitions and elementary facts about secant varieties and rank. The goal is to introduce notation and perhaps help the unexperienced reader to become more familiar with these notions. We also fix some standard notation for reductive algebraic groups, their Lie algebras and their representations.

Throughout the paper we use the following notation. The letter $\mathbb{X}$ is always used for a projective variety and X denotes the affine cone over it. For any subset $S \subset V$ we denote by $\langle S \rangle \subset V$ the span of S. For any subset $\mathbb{S} \in \mathbb{P}(V)$ we denote by $\langle S \rangle \subset V$ the span of the cone S of $\mathbb{S}$ in V. For any non-zero vector $v \in V$ we denote by $[v]$ the class of it in $\mathbb{P}(V)$. If $v = 0$, we set $[v] := 0$. We use $\overline{S}$ to denote the Zariski closure of a set S in a given algebraic variety.

2.1. Secant varieties and rank: general definitions

Let $\mathbb{X} \subset \mathbb{P}$ be an algebraic variety and $X \subset V$ denote the affine cone over $\mathbb{X}$. We denote by $\mathbb{P}\langle\mathbb{X}\rangle = \mathbb{P}(\langle X\rangle)$ the corresponding projective subspace of $\mathbb{P}$. We say that $\mathbb{X}$ spans $\mathbb{P}$ if $\mathbb{P}\langle\mathbb{X}\rangle = \mathbb{P}$; this is equivalent to the requirement that X contains a basis of V. Assume that this is the case. Then every point in V can be written as a linear combination of points in X. This allows us to define the notion of rank already given in the introduction: the rank of $[\psi] \in \mathbb{P}$ with respect to $\mathbb{X}$ is the minimal number of elements of X necessary to express ψ as a linear combination. Thus, the space $\mathbb{P}$ is partitioned into the rank subsets,

$$\mathbb{P} = \mathbb{X}_1 \sqcup \mathbb{X}_2 \sqcup \dots .$$

Since $\mathbb{X}$ spans $\mathbb{P}$, we have $\mathbb{X}_r = \emptyset$ for $r > \dim \mathbb{P}$.

The following properties of the varieties $\mathbb{X}_r$ hold:

(i) $\mathbb{X}_1 = \mathbb{X}$.
(ii) There exists a maximal $r_m \in \{1, \dots, \dim V\}$, such that $\mathbb{X}_{r_m} \neq \emptyset$ and $\mathbb{X}_r = \emptyset$ for $r > r_m$.
(iii) If $r \in \{1, \dots, r_m\}$, then $\mathbb{X}_r \neq \emptyset$.
(iv) The projective space $\mathbb{P}$ can be written as a disjoint union $\mathbb{P} = \mathbb{X}_1 \sqcup \dots \sqcup \mathbb{X}_{r_m}$.

Let $r \in \{2, \dots, r_m\}$. The subset $\mathbb{X}_r \subset \mathbb{P}$ is not closed, because we have $\mathbb{X} \subset \overline{\mathbb{X}_r}$ and $\mathbb{X} \not\subset \mathbb{X}_r$. The r-th secant variety of $\mathbb{X}$ is defined as

$$\sigma_r(\mathbb{X}) = \overline{\bigsqcup_{s \leq r} \mathbb{X}_s} \subset \mathbb{P}.$$

It can also be written as

$$\sigma_r(\mathbb{X}) = \overline{\bigcup_{x_1, \dots, x_r \in \mathbb{X}} \mathbb{P}_{x_1 \dots x_r}},$$

where $\mathbb{P}_{x_1 \dots x_r}$ stands for the projective subspace of $\mathbb{P}$ spanned by the points $x_1, \dots, x_r$.

The following properties of the secant varieties $\sigma_r(\mathbb{X})$ hold:

(i) $\sigma_1(\mathbb{X}) = \mathbb{X}_1 = \mathbb{X}$.
(ii) $\sigma_r(\mathbb{X}) \subset \sigma_{r+1}(\mathbb{X})$.
(iii) If $\mathbb{X}$ is irreducible, then $\sigma_r(\mathbb{X})$ is also irreducible.
(iv) There exists a minimal number $r_g \in \{1, \dots, r_m\}$ such that $\sigma_{r_g}(\mathbb{X}) = \mathbb{P}$ and $\sigma_{r_g - 1}(\mathbb{X}) \neq \mathbb{P}$.
(v) For $r \in \{1, \dots, r_g\}$ the rank subset $\mathbb{X}_r$ is dense in $\sigma_r(\mathbb{X})$, i.e. we have $\sigma_r(\mathbb{X}) = \overline{\mathbb{X}_r}$.

Definition 2.1. The number r_g from part (iv) above is called the *typical rank* of $\mathbb{P}$ with respect to $\mathbb{X}$.

Let $[\psi] \in \mathbb{P}$. The border rank of $[\psi]$ with respect to $\mathbb{X}$ is defined as

$$\underline{\mathrm{rk}}[\psi] := \underline{\mathrm{rk}}_{\mathbb{X}}[\psi] := \min\{r \in \mathbb{N} : [\psi] \in \overline{\mathbb{X}_r}\}.$$

Definition 2.2. Points $[\psi] \in \mathbb{P}$, for which $\mathrm{rk}[\psi] \neq \underline{\mathrm{rk}}[\psi]$, are called *exceptional*.

Clearly, $\mathrm{rk}[\psi] \geq \underline{\mathrm{rk}}[\psi]$ and $[\psi]$ is exceptional exactly when $\underline{\mathrm{rk}}[\psi] < \mathrm{rk}[\psi]$. So, exceptional points are points which can be approximated by points of lower rank. Also, we have

$$\underline{\mathrm{rk}}[\psi] = \min\{r \in \mathbb{N} : [\psi] \in \sigma_r(\mathbb{X})\}.$$

This leads us to the next definition.

Definition 2.3. (i) The secant variety $\sigma_r(\mathbb{X})$ is called *r-discontinuous* if it contains an exceptional vector of border rank r, and *rs-continuous* if it does not.

(ii) The embedding $\mathbb{X} \subset \mathbb{P}$ is called *rs-continuous*, if all secant varieties $\sigma_r(\mathbb{X})$ are rs-continuous. Equivalently, $\mathbb{X} \subset \mathbb{P}$ is rs-continuous if rk is a lower semi-continuous function on $\mathbb{P}$. We say that $\mathbb{X} \subset \mathbb{P}$ is *r-discontinuous* if $\mathbb{X} \subset \mathbb{P}$ is not rs-continuous.

(iii) The embedding $\mathbb{X} \subset \mathbb{P}$ is called i*-continuous*, if $\sigma_i(\mathbb{X})$ is rs-continuous. The embedding $\mathbb{X} \subset \mathbb{P}$ is called i*-discontinuous*, if $\sigma_i(\mathbb{X})$ is r-discontinuous and $\sigma_s(\mathbb{X})$ is rs-continuous for $s < i$.

We record another list of simple statements, which are derived immediately from the above definitions.

(1) The secant variety $\sigma_r(\mathbb{X})$ is r-discontinuous if and only if $\sigma_r(\mathbb{X}) \neq \mathbb{X}_1 \sqcup \mathbb{X}_2 \sqcup \ldots \sqcup \mathbb{X}_r$.

(2) The embedding $\mathbb{X} \subset \mathbb{P}$ is r-discontinuous if and only if $\mathrm{rk}_{\mathbb{X}} \neq \underline{\mathrm{rk}}_{\mathbb{X}}$.

We denote by $X_r \subset V$ the cone over $\mathbb{X}_r$ without 0 and by $\sigma_r(X) \subset V$ the cone over $\sigma_r(\mathbb{X})$ with 0. We denote by $T\mathbb{X}$ the union over all points of $\mathbb{X}$ of tangent spaces to X in $\mathbb{P}$. We say that $\sigma_2(\mathbb{X})$ is *nondegenerate* if $\dim \sigma_2(\mathbb{X}) = 2 \dim \mathbb{X} + 1$. We say that $T\mathbb{X}$ is *nondegenerate* if $\dim T\mathbb{X} = 2 \dim \mathbb{X}$. The following propositions provide some insight to the rs-continuity/r-discontinuity of varieties $\mathbb{X} \subset \mathbb{P}$, without use of group actions.

Proposition 2.1 ([12, Corollary 4]). *If $\mathbb{X}$ is a smooth projective variety, then precisely one of the following must hold:*

(i) $\dim T\mathbb{X} = 2 \dim \mathbb{X}$ *and* $\dim \sigma_2(\mathbb{X}) = 2 \dim \mathbb{X} + 1$, *or*

(ii) $T\mathbb{X} = \sigma_2(\mathbb{X})$.

Proposition 2.2 (Zak's theorem on linear normality [13,32], see also [22, Theorem 1.1]). *Assume that $\mathbb{X}$ is smooth, nondegenerate and $\mathbb{P} \neq \sigma_2(\mathbb{X})$. Then* $\mathrm{codim}_{\mathbb{P}} \mathbb{X} \geq \frac{\dim \mathbb{X}}{2} + 2$.

Corollary 2.3. *Assume that $\mathbb{X}$ is smooth, nondegenerate and* $\mathrm{codim}_{\mathbb{P}} \mathbb{X} < \frac{\dim \mathbb{X}}{2} + 2$. *Then $\mathbb{X}$ is rs-continuous.*

Proposition 2.4 (Landsberg–Roberts [22, Theorem 10.3], [29, Introduction]). *Assume that $\mathbb{X}$ is smooth and* $\mathrm{codim}_{\mathbb{P}} \mathbb{X} > \binom{\dim X+1}{2}$. *Then $\sigma_2(\mathbb{X})$ is nondegenerate.*

Some representations satisfy the conditions of Corollary 2.3 and thus their rs-continuity is checked by the general machinery. Nevertheless, to study all possible representations we need more methods. The general methods we employ to decide rs-continuity for nonfundamental representations are independent of codimension of the variety or degeneracy of its secant variety.

2.2. Irreducible representations of reductive groups

Here we fix some notation concerning semisimple or reductive algebraic groups and their representations. All notions from this theory used by us can be found in [14] (the notation, however, may differ).

Let G be a connected semisimple algebraic group over $\mathbb{F}$ and $\mathfrak{g}$ be its Lie algebra. We assume that the semisimple part G' of G is simply connected. Let $B \subset G$ be a Borel subgroup and $T \subset B$ be a maximal torus. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ denote the respective subalgebras of $\mathfrak{g}$. Let $\Lambda \subset \mathfrak{h}^*$ be the integral weight lattice and $\Delta \subset \Lambda$ the root system. Let $\Delta = \Delta^+ \sqcup \Delta^-$ be the partition of the root system into positive and negative roots corresponding to the Borel subgroup B. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be the basis of simple roots in Δ^+.

The Cartan-Killing form of $\mathfrak{g}$ determines a scalar product $(\cdot, \cdot)$ on $\mathfrak{h}^*$. This scalar product $(\cdot, \cdot)$ defines a Dynkin diagram (graph) Dyn, whose vertices are labeled by the simple roots $\alpha_1, \dots, \alpha_\ell \in \Pi$. Also Π and the scalar product define the dominant Weyl chamber Λ and the monoid of dominant weights $\Lambda^+ \subset \Lambda$, generated by the fundamental weights $\pi_1, \dots, \pi_\ell$. Hence $\lambda \in \Lambda^+$ defines a function $f_\lambda : \Pi \to \mathbb{Z}_{\geq 0}$ such that $\lambda = f_\lambda(\alpha_1)\pi_1 + \cdots + f_\lambda(\alpha_\ell)\pi_\ell$. This defines a one-to-one correspondence between the set Λ^+ of dominant weights and functions from the set of vertices of Dyn to the non-negative integers. We put

$$h(\lambda) := f_\lambda(\alpha_1) + \cdots + f_\lambda(\alpha_\ell).$$

The set Λ^+ is also in a one-to-one correspondence with the set of isomorphism classes of simple finite-dimensional G-modules. We denote by $V(\lambda)$ the irreducible representation of G corresponding to $\lambda \in \Lambda^+$ (λ is a highest weight of $V(\lambda)$ and $V(\lambda)$ contains a unique up to scaling vector v^λ of weight λ). Set $\mathbb{P}(\lambda) := \mathbb{P}(V(\lambda))$. We denote by $X(\lambda)$ and $\mathbb{X}(\lambda)$ the orbits of v^λ and $[v^\lambda]$ in $V(\lambda)$ and $\mathbb{P}(\lambda)$ respectively. Note that $\mathbb{X}(\lambda)$ is the unique closed G-orbit on $\mathbb{P}(\lambda)$.

Any subdiagram of a Dynkin diagram is again a Dynkin diagram. Thus, by chopping down some vertices (along with the adjacent edges) we obtain a new diagram $\underline{Dyn}$, which corresponds to a semisimple Levi subgroup $\underline{G} \subset G$. The restriction of f_λ to $\underline{Dyn}$ defines a simple representation $\underline{V}$ of the group $\underline{G}$.

Definition 2.4. We say that a $\underline{G}$-representation $\underline{V}$ is a *chopping* of a G-representation V, if $\underline{V}$ is obtained from V via the above construction.

Remark 2.1. Note that a chopping of a fundamental representation is either fundamental, or trivial one-dimensional.

The Dynkin diagram determines the Weyl group $\mathcal{W}$. We denote by w_0 the longest element of $\mathcal{W}$ with respect to the Bruhat order. The weights of the form $w\lambda$, with $w \in \mathcal{W}$, are called *extreme weights* of the module $V(\lambda)$ and the corresponding weight vectors are called extreme weight vectors. For any $w \in \mathcal{W}$ we denote by $v^{w\lambda}$ the unique up to scaling vector of weight $w\lambda$. The weight $w_0\lambda$ is called *the lowest weight* of $V(\lambda)$.

For $\lambda_1, \lambda_2 \in \Lambda^+$, G-module $V(\lambda_1) \otimes V(\lambda_2)$ contains a unique up to scaling vector of weight $\lambda_1 + \lambda_2$. This vector is contained in a simple G-submodule, which is isomorphic to $V(\lambda_1 + \lambda_2)$. We call this submodule the *Cartan component* of $V(\lambda_1) \otimes V(\lambda_2)$. It is well known that the Cartan component does not depend on a choice of Borel subgroup $B \subset G$.

Let us fix $\lambda \in \Lambda^+$ and put $V = V(\lambda)$ and $\mathbb{P} = \mathbb{P}(\lambda) := \mathbb{P}(V(\lambda))$. The group G acts on the projective space $\mathbb{P}$ and has a unique closed orbit therein, namely, the orbit through the highest weight line, to be denoted by $\mathbb{X} = \mathbb{X}(\lambda) := G[v^\lambda]$. We have $\mathbb{X} = G/P$, where P denotes the stabilizer of $[v^\lambda] \in \mathbb{P}$ in G. This P is a standard parabolic subgroup, i.e. a closed subgroup of G containing the fixed Borel subgroup B. The cosets of G by parabolic subgroups are called the flag varieties of G. Thus we have an equivariantly embedded flag variety $\mathbb{X} = G/P \subset \mathbb{P}$. In fact, all equivariantly embedded homogeneous projective varieties are obtained in this fashion. Note, that the variety $\mathbb{X}$ is the set of highest weight vectors with respect to all possible choices of Borel subgroups $B \subset G$.

The irreducibility of V implies that $\mathbb{X}$ spans $\mathbb{P}$. Hence, we have well defined rank and border rank functions on $\mathbb{P}$ with respect to $\mathbb{X}$, as well as secant varieties $\sigma_r(\mathbb{X}) \subset \mathbb{P}$. Since the group G acts on V by invertible linear transformations, it follows immediately that rank and border rank are G-invariant functions. Hence, the rank sets $\mathbb{X}_r$ and the secant varieties $\sigma_r(\mathbb{X})$ are preserved by G.

Definition 2.5. Let $V = V(\lambda)$, with $\lambda \in \Lambda^+$, be an irreducible representation of a reductive linear algebraic group G. Let $\mathbb{X} = \mathbb{X}(\lambda)$ be the unique closed G-orbit in $\mathbb{P} = \mathbb{P}(\lambda)$. The G-module V is called *rs-continuous* (resp. *r-discontinuous*, *i-continuous*, *i-discontinuous*), if the variety $\mathbb{X} \subset \mathbb{P}$ is rs-continuous (resp. r-discontinuous, i-continuous, i-discontinuous).

Our goal is to classify all rs-continuous irreducible representations of semisimple algebraic groups.

Remark 2.2. In some of our constructions we consider reductive groups, rather than semisimple groups, just because this simplifies some steps. The actions of G and G' (the commutant of G) on $\mathbb{P}$ coincide and we are concerned with properties of the embedding $\mathbb{X} \subset \mathbb{P}$. Thus the classification of rs-continuous representations of reductive groups can be easily obtained from the one for semisimple groups.

Below we assume that $\lambda \neq 0$ (this corresponds to the inequality $\dim V \geq 2,\ V = V(\lambda)$, assumed in Theorem 1.1).

We denote by A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 the simple simply connected algebraic groups with the corresponding Dynkin diagrams. We denote by $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$, $\mathfrak{d}_n$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$ the corresponding Lie algebras.

3. Plan of proof of Theorem 1.1

Theorem 1.1 follows from several propositions and theorems proved throughout the article. Thus the plan explains the role of different parts of this text.

In Proposition 4.6, we prove that, if an irreducible representation $V(\lambda)$ with highest weight λ is rs-continuous, then $h(\lambda) < 3$. In Proposition 4.8, we classify all irreducible rs-continuous modules $V(\lambda)$ with $h(\lambda) = 2$.

The rs-continuous fundamental representations (i.e. irreducible G-modules $V(\lambda)$ with $h(\lambda) = 1$) are classified in Theorems 6.1 and 7.1. This is done in a case by case study. We consider separately representations of classical groups (Theorem 6.1), and representations of exceptional groups (Theorem 7.1). This completes the proof of Theorem 1.1.

Let us say a few words about the proofs of Theorems 6.1 and 7.1. The rs-continuous representations are considered individually and for each case we provide specific arguments. Most representations are r-discontinuous and thus we have to check r-discontinuity of a huge amount of cases. The number of cases to be considered is greatly reduced by Proposition 5.1, where we show that, if a $\underline{G}$-representation $\underline{V}$ is a chopping of a G-representation V and $\underline{V}$ is r-discontinuous, then V is also r-discontinuous. Thus, it suffices to check r-discontinuity directly for only few basic cases, then we are able to deduce r-discontinuity for most of the fundamental representations.

4. Non-fundamental rs-continuous modules

The goal of this section is to classify all non-fundamental rs-continuous modules. First of all we provide in Lemma 4.3 of Sect. 4.1 a way to construct exceptional vectors in $V(\lambda)$. Using this construction we show in Proposition 4.6 of Sect. 4.2 that, if a G-module $V(\lambda)$ is 2-continuous, then it is either a fundamental G-module (i.e. $h(\lambda) = 1$) or is a Cartan component in a tensor product of two fundamental G-modules (i.e. $h(\lambda) = 2$). Further on, in Proposition 4.7, we shall show that, in the latter situation, each of the fundamental modules satisfies a very strict condition, related to the notion of HW-density introduced in Definition 4.2. Using the explicit description of all HW-dense modules which we present in Corollary 4.5, we are able to complete in Proposition 4.8 a classification of non-fundamental rs-continuous G-modules $V(\lambda)$ (i.e. modules $V(\lambda)$ such that $h(\lambda) = 2$).

4.1. The varieties $\sigma_2(\mathbb{X}(\lambda))$ and $\mathbb{X}_2(\lambda)$

In some sense, in this article we study the difference between the variety $\sigma_2(\mathbb{X}(\lambda))$ and its open subset $\mathbb{X}_2(\lambda)$. Here we collect some of their basic features. First we recall that, if the generic rank of $V(\lambda)$ is greater than 1 (i.e. if $\mathbb{P}(\lambda) \neq \mathbb{X}(\lambda)$), then

$$\sigma_2(X(\lambda)) = \overline{X_2(\lambda)}.$$

We provide an explicit description of the elements of $\mathbb{X}_2(\lambda)$ (Lemma 4.1) and exhibit a set of elements of $\sigma_2(\mathbb{X}(\lambda))$, which tend to be exceptional (Lemma 4.3).

Lemma 4.1. *Let $V(\lambda)$ be an irreducible representation of a reductive group G. Then*

(a) any pair $([v_1], [v_2]) \in \mathbb{X}(\lambda) \times \mathbb{X}(\lambda)$ is G-conjugate to a pair $([v^\lambda], [v^{w\lambda}])$ for some $w \in \mathcal{W}$,

(b) any element of $\mathbb{X}_2(\lambda)$ *is conjugate to* $[v^\lambda + v^{w\lambda}]$ *for some* $w \in \mathcal{W}$.

Proof. Fix $v_1, v_2 \in X(\lambda)$ such that $[v_1] \neq [v_2]$. Let B_1, B_2 be Borel subgroups of G such that v_1, v_2 are the corresponding to B_1, B_2 highest weight vectors. It is known that $B_1 \cap B_2$ contains a maximal torus T_{12} of G and there exists $w \in \mathcal{W}_{12} := N_G(T_{12})/T_{12}$ such that $B_1 = wB_2$. Thus $([v_1], [v_2])$ is conjugate to $([v^\lambda], [v^{w\lambda}])$ for some $w \in \mathcal{W}$. This completes the proof of part a).

To prove part b) we observe that any element of $X_2(\lambda)$ is a sum $v_1 + v_2$ for some $v_1, v_2 \in X(\lambda)$ such that $v_1 \neq v_2$. According to part a) the pair $([v_1], [v_2])$ is conjugate to $([v^\lambda], [v^{w\lambda}])$ for some $w \in \mathcal{W}$. Thus $[v_1 + v_2]$ is conjugate to $[av^\lambda + bv^{w\lambda}]$ for some non-zero $a, b \in \mathbb{F}$. As $[v_1] \neq [v_2]$, $\lambda \neq w\lambda$. Hence $[v^\lambda + v^{w\lambda}]$ is conjugate to $[av^\lambda + bv^{w\lambda}]$ for any non-zero $a, b \in \mathbb{F}$. This completes the proof of b). ◻

Corollary 4.2. *Let* w_0 *be the longest Weyl group element. The orbit* $G[v^\lambda + v^{w_0\lambda}]$ *is open in both varieties* $\sigma_2(\mathbb{X}(\lambda))$ *and* $\mathbb{X}_2(\lambda)$ *(see also [32, Ch. III, Thm 1.4]).*

Lemma 4.3. *Fix* $x \in X(\lambda)$ *and* $t \in \mathfrak{g}$. *Then* $[x + tx] \in \sigma_2(\mathbb{X}(\lambda))$.

Proof. By definition $\mathbb{X}_2(\lambda) \cup \mathbb{X}(\lambda)$ is the union of lines going through pairs of points of $\mathbb{X}(\lambda)$. Thus the tangent space $T_x X(\lambda) \subset V$ to $X(\lambda)$ in x belongs to $\overline{X_2(\lambda)} = \sigma_2(X(\lambda))$. On the other hand, $x + tx$ belongs to $T_x X(\lambda)$ as tx is tangent to $X(\lambda)$. This completes the proof. ◻

4.2. HW-density and 2-continuity

In this subsection, we analyze the notion of 2-continuity via the notion of HW-density given in Definition 4.2. This analysis allows to find out all rs-continuous modules which are not fundamental. Notice that 2-continuity is, a priori, weaker than rs-continuity, but is much simpler to check. A posteriori, it turns out that rs-continuity and 2-continuity are equivalent for the class of homogeneous projective varieties considered in this paper.

We proceed in the following way. We prove that, if an irreducible G-module V is 2-continuous, then it is either a fundamental module of G (i.e. only one mark of the highest weight of V is distinct from zero and this mark equals 1) or is a Cartan component of the tensor product of two fundamental modules of G, see Proposition 4.6. In the latter case we prove that both fundamental modules in the product have to be HW-dense, see Proposition 4.7. It turns out that HW-density is a very strict condition as we show in Corollary 4.5. This result leads to Proposition 4.8, which lists all 2-continuous G-modules, which are not fundamental.

We start with the definition of HW-density followed by the statements of the results. The proofs of these results are given below until the end of the section.

Definition 4.1. Let $\mathbb{X}$ be a smooth subvariety of a projective space $\mathbb{P}$. We say that $\mathbb{X}$ is *HW-dense*, if for any point $x_1 \in \mathbb{X}$ there exists an open subset U of the tangent space to $\mathbb{X}$ at x_1 such that for all $v \in U$ there exists $x_2 \in \mathbb{X}$ such that $v \in \langle x_1, x_2 \rangle$.

Definition 4.2. We say that a simple G-module $V(\lambda)$ is *HW-dense*, if $\mathbb{X}(\lambda)$ is HW-dense in $\mathbb{P}(\lambda)$.

We shall prove below in this subsection the following criterion of HW-density.

Lemma 4.4. *The G-module $V(\lambda)$ is HW-dense if and only if one of the following equivalent conditions holds:*

(a) The set $X(\lambda)$ of highest weight vectors is dense in $V(\lambda)$.
(b) All non-zero vectors of V are highest weight vectors with respect to some choice of a Borel subgroup of G.
(c) G acts transitively on $\mathbb{P}(V)$.

Corollary 4.5. *Let V be an effective fundamental HW-dense G-module. Then (G, V) is isomorphic to $(Sp(V), V)$, $(SL(V), V)$ or $(SL(V), V^*)$.*

Now, we formulate Propositions 4.6, 4.7 announced at the beginning of Sect. 4.

Proposition 4.6. *Assume that $V(\lambda)$ is rs-continuous. Then $h(\lambda) < 3$.*

Proposition 4.7. *Let $\lambda_1, \lambda_2 \in \Lambda$ be non-zero weights. Assume that $V(\lambda_1 + \lambda_2)$ is 2-continuous. Then both $V(\lambda_1)$, $V(\lambda_2)$ are HW-dense.*

Proofs of Propositions 4.6, 4.7 are presented below in this subsection. Corollary 4.5 and Proposition 4.7 immediately imply the following proposition.

Proposition 4.8. *Assume that $V(\lambda)$ is an effective 2-continuous module and $h(\lambda) =$ 2. Then $(G, V(\lambda))$ appears in the following list:*

(1) $(SL(V_1) \times SL(V_2), V_1 \otimes V_2)$; (2) $(SL(V_1) \times Sp(V_2), V_1 \otimes V_2)$;
(3) $(Sp(V_1) \times Sp(V_2), V_1 \otimes V_2)$;
(4) $(SL(V), S^2 V)$; (5) $(SL(V), \mathfrak{sl}(V))$; (6) $(Sp(V), \mathfrak{sp}(V)) \cong (Sp(V), S^2 V)$

(here $\mathfrak{sl}(V)$, $\mathfrak{sp}(V)$ denote the adjoint modules of the corresponding groups).

All modules listed in Proposition 4.8 are rs-continuous. Cases (1) and (4) of Lemma 4.8 are known to be rs-continuous. Cases (2) and (3) have the same secant varieties as case 1), and hence are also rs-continuous. Cases (5) and (6) are rs-continuous due to [19]. Therefore Proposition 4.8 explicitly lists all non-fundamental rs-continuous modules.

The rest of the current subsection is dedicated to the proofs of Propositions 4.6, 4.7 and Lemma 4.4. We need the following lemma for the $GL(V_1) \times GL(V_2) \times GL(V_3)$-module $V_1 \otimes V_2 \otimes V_3$, where V_1, V_2, V_3 are finite-dimensional vector spaces.

Lemma 4.9. *Let x_i, y_i be linearly independent vectors in V_i for $i = 1, 2, 3$. Then*

$$\begin{aligned} T := x_1 \otimes x_2 \otimes x_3 + y_1 \otimes x_2 \otimes x_3 + x_1 \otimes y_2 \otimes x_3 \\ + x_1 \otimes x_2 \otimes y_3 \neq v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3 \end{aligned}$$

for all $v_i, w_i \in V_i$ $(i = 1, 2, 3)$. In other words, $\operatorname{rk} T > 2$.

Proof. Assume on the contrary that

$$T = v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3$$

for some $v_i, w_i \in V_i, i = 1, 2, 3$. We have $V_1 \otimes V_2 \otimes V_3 \cong \mathrm{Hom}(V_2^* \otimes V_3^*, V_1)$. Therefore for any $x \in V_1 \otimes V_2 \otimes V_3$ we can define $\mathrm{Im}_1(x) \subset V_1$ as the image of the corresponding homomorphism from $\mathrm{Hom}(V_2^* \otimes V_3^*, V_1)$. Similarly we define $\mathrm{Im}_2(x)$ and $\mathrm{Im}_3(x)$. We have

$$\mathrm{Im}_i(T) = \langle x_i, y_i \rangle, i = 1, 2, 3.$$

On the other hand, if $v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3 \neq 0$,

$$\mathrm{Im}_i(v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3) \subset \langle v_i, w_i \rangle \ (i = 1, 2, 3).$$

Therefore, $\langle x_i, y_i \rangle = \langle v_i, w_i \rangle, i = 1, 2, 3$. Hence, without loss of generality, we may assume that

$$V_i = \langle x_i, y_i \rangle = \langle v_i, w_i \rangle \ (i = 1, 2, 3),$$

i.e. that $\dim V_i = 2$ $(i = 1, 2, 3)$. For two-dimensional spaces V_1, V_2, V_3 this lemma is well known; see e.g. [23]. □

Note that for $(G, V(\lambda)) = (SL(V_1) \times SL(V_2) \times SL(V_3), V_1 \otimes V_2 \otimes V_3)$ we have $h(\lambda) = 3$. Lemma 4.9 is essentially a particular case of Proposition 4.6, which we shall use to prove the general case.

Proof of Proposition 4.6. Assume on the contrary that $h(\lambda) \geq 3$. Then there exist non-zero weights $\lambda_1, \lambda_2, \lambda_3 \in \Lambda^+$ such that $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Then

$$v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} \in V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$$

is a highest weight vector of weight $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Therefore the smallest G-submodule of $V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$ containing $v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3}$ is isomorphic to $V(\lambda)$. We identify $V(\lambda)$ with this submodule of $V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$ and set

$$v^{\lambda} := v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3}.$$

By Lemma 4.3 we have

$$v^{\lambda} + tv^{\lambda} \in \overline{X_2(\lambda)} (= \sigma_2(X(\lambda)))$$

for any $t \in \mathfrak{g}$. Therefore

$$\underline{\mathrm{rk}}_{\mathbb{X}(\lambda)}(v^{\lambda} + tv^{\lambda}) \leq 2 \tag{2}$$

for any $t \in \mathfrak{g}$. By the Leibniz rule we have

$$\begin{aligned} T_{\lambda} := v^{\lambda} + tv^{\lambda} = v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} + tv^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} + v^{\lambda_1} \otimes tv^{\lambda_2} \otimes v^{\lambda_3} \\ + v^{\lambda_1} \otimes v^{\lambda_2} \otimes tv^{\lambda_3} \end{aligned}$$

(note that T_{λ} is of the form T of Lemma 4.9). We claim that

$$v^{\lambda} + tv^{\lambda} \notin X_2(\lambda) \cup X(\lambda) \cup \{0\}$$

for all $t \in U$ from some open subset U of $\mathfrak{g}$. As $\lambda_i \neq 0$, there exists some open subset $U \subset \mathfrak{g}$ such that $[tv^{\lambda_i}] \neq [v^{\lambda_i}]$ $(i = 1, 2, 3)$ for all $t \in U$. We fix $t \in U$. We claim that

$$\mathrm{rk}(v^\lambda + tv^\lambda) \geq 3. \tag{3}$$

Assume on the contrary that $v^\lambda + tv^\lambda \in X_2(\lambda) \cup X(\lambda) \cup 0$, then

$$v^\lambda + tv^\lambda = g_1(v^{\lambda_1}) \otimes g_1(v^{\lambda_2}) \otimes g_1(v^{\lambda_3}) + g_2(v^{\lambda_1}) \otimes g_2(v^{\lambda_2}) \otimes g_2(v^{\lambda_3})$$

for some $g_1, g_2 \in G$ and thus

$$T_\lambda = v^\lambda + tv^\lambda = v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3$$

for some $v_1, v_2, v_3, w_1, w_2, w_3 \in V(\lambda)$. This contradicts the statement of Lemma 4.9. Comparing (2) and (3) we see that $v^\lambda + tv^\lambda$ is an exceptional vector of $V(\lambda)$ and thus $V(\lambda)$ is 2-discontinuous. □

Proof of Proposition 4.7. We use notation analogous to the one in Proposition 4.6. Fix $\lambda := \lambda_1 + \lambda_2$. Set

$$v^\lambda := v^{\lambda_1} \otimes v^{\lambda_2} \in V(\lambda_1) \otimes V(\lambda_2).$$

This defines a canonical embedding $V(\lambda) \to V(\lambda_1) \otimes V(\lambda_2)$. As $\lambda_i \neq 0$, there exists some open subset $U \subset \mathfrak{g}$ such that $[tv^{\lambda_i}] \neq [v^{\lambda_i}]$ $(i = 1, 2)$ for all $t \in U$. We fix $t \in U$. Repeating the argument preceding (2) we deduce that (2) holds in the present notation. Hence, since $V(\lambda)$ is 2-continuous we have

$$v^\lambda + tv^\lambda = g_1 v^\lambda + g_2 v^\lambda \quad \text{or} \quad v^\lambda + tv^\lambda = g_1 v^\lambda$$

for some $g_1, g_2 \in G$. Then

$$\mathrm{Im}_i(v^\lambda + tv^\lambda) = \langle v^{\lambda_i}, tv^{\lambda_i} \rangle \ (i = 1, 2),$$

(for the definition of Im_i see proof of Lemma 4.9) and, if $g_1 v^\lambda + g_2 v^\lambda \neq 0$,

$$\mathrm{Im}_i(g_1 v^\lambda + g_2 v^\lambda) = \langle g_1 v^{\lambda_i}, g_2 v^{\lambda_i} \rangle \ (i = 1, 2) \quad \text{or} \quad \mathrm{Im}_i(g_1 v^\lambda + g_2 v^\lambda) = \langle g_1 v^{\lambda_i} \rangle.$$

Hence $g_1 v^{\lambda_i}, g_2 v^{\lambda_i} \in \langle v^{\lambda_i}, tv^{\lambda_i} \rangle$ and either

$$[g_1 v^{\lambda_i}] \neq [v^{\lambda_i}] \quad \text{or} \quad [g_2 v^{\lambda_i}] \neq [v^{\lambda_i}] \ (i = 1, 2).$$

Therefore both $V(\lambda_1)$ and $V(\lambda_2)$ are HW-dense. This completes the proof. □

To prove Lemma 4.4 we need two technical lemmas. The first gives a reformulation of Definition 4.2.

Lemma 4.10. *A simple G-module $V(\lambda)$ is HW-dense if and only if there exists an open subset $U \subset \mathfrak{g}$ such that, for all $t \in U$, there exists an element*

$$v \in X(\lambda) \cap \langle v^\lambda, tv^\lambda \rangle$$

such that $[v] \neq [v^\lambda]$ (note that if such an element v exists, then $[tv^\lambda] \neq [v^\lambda]$).

Proof. This is a fairy easy exercise for Lie algebras-Lie groups formalism. We omit it. $\square$

Lemma 4.11. *Fix $v_1 \in X(\lambda)$. Assume that there exists $v_2 \in X(\lambda)$ such that $v_2 \in \langle v_1, \mathfrak{g}v_1 \rangle$. Then all non-zero vectors of $\langle v_1, v_2 \rangle$ belong to $X(\lambda)$.*

Proof. Without loss of generality we assume that $[v_1] \neq [v_2]$. Then, by Lemma 4.1a), the pair (v_1, v_2) is conjugate to the pair $(v^\lambda, v^{w\lambda})$ for some $w \in \mathcal{W}$ and thus we can assume that

$$v_1 = v^\lambda \quad \text{and} \quad v_2 = v^{w\lambda}$$

for the fixed maximal torus $T \in G$. The space $\mathfrak{g}v_1$ is clearly T-invariant and the weights of this space form a subset of the set $\lambda + \Delta$ (this is a point-wise sum). As $v^{w\lambda} \in \mathfrak{g}v^\lambda$, we have

$$w\lambda = \lambda + \beta \quad \text{for some} \quad \beta \in \Delta$$

(note that $w\lambda \neq \lambda$ as $[v_1] \neq [v_2]$). Let $SL_2(\beta)$ be the T-stable SL_2-subgroup corresponding to the root $\beta \in \Delta$. Then the space $\langle v^\lambda, v^{w\lambda} \rangle$ is a two-dimensional simple $SL_2(\beta)$-module and thus any two non-zero elements of $\langle v_1, v_2 \rangle$ are $SL_2(\beta)$-conjugate, and hence G-conjugate. Therefore all non-zero elements of $\langle v_1, v_2 \rangle$ belong to $X(\lambda)$. $\square$

Proof of Lemma 4.4. The equivalence of conditions (a), (b), (c) is clear. It is also immediate to verify that each of these conditions implies HW-density. It remains to show that, if the module $V(\lambda)$ is HW-dense, then it satisfies condition (a).

Assume that $V(\lambda)$ is HW-dense. Then according to Lemma 4.10 and Lemma 4.11 there exists an open subset $U \subset \mathfrak{g}$ such that for any non-zero $a \in \mathbb{F}$ and any $t \in U$ we have

$$v^\lambda + atv^\lambda \in X(\lambda),$$

i.e. we have that $X(\lambda) \cap \langle v^\lambda, \mathfrak{g}v^\lambda \rangle$ is a dense subset of $\langle v^\lambda, \mathfrak{g}v^\lambda \rangle$. Note that $v^\lambda \in \mathfrak{g}v^\lambda$, as $\lambda \neq 0$, and hence

$$\langle v^\lambda, \mathfrak{g}v^\lambda \rangle = \langle \mathfrak{g}v^\lambda \rangle.$$

We have

$$\dim X(\lambda) = \dim Gv^\lambda = \dim \mathfrak{g}v^\lambda$$

and therefore

$$\overline{X(\lambda)} = \mathfrak{g}v^\lambda.$$

On the other hand

$$V(\lambda) = \langle X(\lambda) \rangle$$

and hence

$$V(\lambda) = \overline{X(\lambda)} = \mathfrak{g}v^\lambda.$$

$\square$

5. Restriction to a Levi subgroup

The main result of this section is Proposition 5.1. This proposition provides a strong sufficient condition for r-discontinuity of a representation. We will apply this proposition to study rs-continuity/r-discontinuity of fundamental representations in the subsequent sections of this article.

Proposition 5.1. *Let V be an irreducible G-module and $\underline{V}$ be a $\underline{G}$-module, which is a chopping of V. If $\underline{V}$ is r-discontinuous, then V is r-discontinuous.*

This proposition is an immediate corollary of Proposition 5.2. To state Proposition 5.2 we need more notation.

Recall that we have fixed Cartan and Borel subgroups $H \subset B \subset G$ and that Π denotes the corresponding set of simple roots. Let $\underline{\Pi} \subset \Pi$ be a subset. Then $\underline{\Delta} = \Delta \cap \langle \underline{\Pi} \rangle$ is a root system having $\underline{\Pi}$ as a set of simple roots and $\underline{\Delta}^{\pm} = \underline{\Delta} \cap \Delta^{\pm}$ as sets of positive and negative roots. Further, let $\underline{\mathfrak{g}} = \mathfrak{h} \oplus (\oplus_{\alpha \in \underline{\Delta}})\mathfrak{g}^{\alpha}$. Then $\underline{\mathfrak{g}}$ is a reductive subalgebra of $\mathfrak{g}$; we call subalgebras of this form (reductive) Levi subalgebras. Let $\underline{G} \subset G$ be the corresponding Levi subgroup. We shall add underline to denote the attributes of $\underline{G}$ with the notational conventions already introduced for G.

Note that G and $\underline{G}$ have a common Cartan subgroup H and hence have the same weight lattice Λ. However, the dominant Weil chambers do not coincide, unless $\underline{\Pi} = \Pi$, a case which is of no use for us. We have an inclusion $\Lambda^+ \subset \underline{\Lambda}^+$, so a weight $\lambda \in \Lambda^+$ can be regarded as a dominant weight for both G and $\underline{G}$. Furthermore, since $\underline{B} = B \cap \underline{G}$, the B-highest weight vectors are also $\underline{B}$-highest weight vectors.

Fix $\lambda \in \Lambda^+$. There is a $\underline{G}$-equivariant inclusion of the corresponding representations

$$\underline{V} = \underline{V}(\lambda) = \mathfrak{U}(\underline{\mathfrak{g}})v^{\lambda} \subset V(\lambda) = V,$$

where v^{λ} denotes the B-highest weight vector in $V(\lambda)$. Let $\underline{\mathbb{X}}$ denote the unique closed $\underline{G}$-orbit in $\mathbb{P}(\underline{V})$ and, as before, let $\mathbb{X}$ denote the unique closed G-orbit in $\mathbb{P}(V)$. We have

$$\underline{\mathbb{X}} = \underline{G}[v^{\lambda}] \subset G[v^{\lambda}] = \mathbb{X}.$$

For points in $\mathbb{P}(\underline{V})$, we have two well defined rank functions $\mathrm{rk}_{\underline{\mathbb{X}}}$ and $\mathrm{rk}_{\mathbb{X}}$ (notice that $\underline{V}$ is a chopping of V according to Definition 2.4). We would like to compare these functions and prove the following.

Proposition 5.2. *Let $\underline{G} \subset G$ and $\underline{V} \subset V$ be as above. If $[\psi] \in \mathbb{P}(\underline{V})$, then* $\mathrm{rk}_{\underline{\mathbb{X}}}[\psi] = \mathrm{rk}_{\mathbb{X}}[\psi]$.

Proof. First, observe that the multiplicity of the $\underline{G}$-module $\underline{V}(\lambda)$ in $V(\lambda)$ is 1. This holds because $\underline{V}(\lambda)$ has a weight vector with weight λ and this weight has multiplicity 1 in $V(\lambda)$. Consequently, there is a well-defined $\underline{G}$-equivariant projection

$$\pi : V \twoheadrightarrow \underline{V}.$$

Let $P \subset G$ be the parabolic subgroup containing B and having $\underline{G}$ as a Levi component; the roots of P are $\Delta^+ \sqcup \underline{\Delta}^-$. Let N_P be the unipotent radical of P; the roots of N_P are $\Delta^+ \backslash \underline{\Delta}^+$. Then N_P acts trivially on $\underline{V}$.

Lemma 5.3. *We have* $\pi(X \cup 0) = \underline{X} \cup 0$.

Proof. Let N_P^- be the nilradical of the parabolic P^- opposite to P, with respect to the given Cartan subgroup H. In other words, N_P^- is the regular unipotent subgroup of N^- with roots $\Delta(N_P^-) = -\Delta(N_P)$. We have

$$X = \overline{Gv^\lambda} = \overline{P^- v^\lambda} = \overline{N_P^-(\underline{G}v^\lambda)} = \overline{N_P^- \underline{X}}.$$

Thus, to prove the lemma it is sufficient to show that for all $g \in N_P^-$ and all $v \in \underline{X}$ we have $\pi(gv) \in \underline{X}$. Let $g \in N_P^-$ and $v \in \underline{X}$. Since the exponential map $\exp : \mathfrak{n}_\mathfrak{p}^- \to N_P^-$ is surjective, we can write $g = \exp(\xi)$ with $\xi \in \mathfrak{n}_\mathfrak{p}^-$. Viewing ξ as an element of $\mathfrak{gl}(V)$ we can write

$$gv = \left(1 + \xi + \frac{1}{2}\xi^2 + \cdots\right) v = v + \xi v + \frac{1}{2}\xi^2 v + \cdots .$$

Let $\underline{V}' = \ker(\pi)$, so that $V = \underline{V} \oplus \underline{V}'$ as $\underline{G}$-modules. Then, for $\xi \in \mathfrak{n}_\mathfrak{p}^-$, we have $\xi(\underline{V}) \subset \underline{V}'$. Hence $\pi(gv) = v$. $\square$

Now, let $[\psi] \in \mathbb{P}(\underline{V})$. The inequality $\mathrm{rk}_{\underline{\mathbb{X}}}[\psi] \geq \mathrm{rk}_{\mathbb{X}}[\psi]$ is immediate. Let $r = \mathrm{rk}[\psi]$ and

$$\psi = v_1 + \cdots + v_r$$

be a minimal expression, with $[v_j] \in \mathbb{X}$. Then we have

$$\psi = \pi(\psi) = \pi(v_1) + \cdots + \pi(v_r)$$

and, according to the above lemma, $[\pi(v_j)] \in \underline{\mathbb{X}} \cup 0$ (this is a set). Hence $\mathrm{rk}_{\underline{\mathbb{X}}}[\psi] \leq \mathrm{rk}_{\mathbb{X}}[\psi]$ and so

$$\mathrm{rk}_{\underline{\mathbb{X}}}[\psi] = \mathrm{rk}_{\mathbb{X}}[\psi].$$

$\square$

6. Fundamental representations (classical groups)

In this subsection we prove Theorem 1.1 for fundamental modules of classical groups, i.e. we prove Theorem 6.1. The result follows directly from Propositions 6.2, 6.4, 6.5 of Sects. 6.1, 6.2, 6.3, where we consider the cases of SL_n, SO_n, Sp_{2n}, respectively.

Theorem 6.1. *Let $V(\lambda)$ be a fundamental module of a simple classical group G. Then $V(\lambda)$ is rs-continuous if and only if the pair $(G, V(\lambda))$ appears in the following table.*

Group G	*Representation V*	*Highest weight of V*
	Classical groups	
SL_n	$\mathbb{F}^n$, $(\mathbb{F}^n)^*$, $(\Lambda^2\mathbb{F}^n)$, $(\Lambda^2\mathbb{F}^n)^*$	$\pi_1, \pi_{n-1}, \pi_2, \pi_{n-2}$
SO_n	$\mathbb{F}^n$, $RSpin_n (n \leq 10)$	$\pi_1, \pi_{\frac{n}{2}} (2 \mid n)$, $\pi_{\frac{n}{2}-1} (2 \mid n), \pi_{\frac{n-1}{2}} (2 \nmid n)$
SP_{2n}	$\mathbb{F}^{2n}$, $\Lambda_0^2\mathbb{F}^{2n}$	π_1, π_2

, (4)

where the notation is the same as in Theorem 1.1.

Moreover, all r-discontinuous fundamental representations of classical groups are 2-discontinuous.

Our approach for classical groups is tensor-based and we often use symmetric/antisymmetric bilinear forms. To prove Theorem 6.1 we also need some sufficient condition of r-discontinuity for representations. Such a condition is provided in Proposition 5.1 of Sect. 5. In a similar way Proposition 5.1 will be very useful in Sect. 7, where we consider the fundamental representations of the exceptional groups.

6.1. $G = SL_n$

Recall that the fundamental representations of $G = SL_n$ are obtained as exterior tensor powers of the natural representation, i.e. $V(\pi_k) = \Lambda^k \mathbb{F}^n$, $k = 1, \ldots, n-1$. Furthermore, we have

$$(\Lambda^k \mathbb{F}^n)^* = \Lambda^{n-k} \mathbb{F}^n$$

as SL_n-modules.

Proposition 6.2. *The fundamental representations of SL_n which are rs-continuous are exactly*

$$\mathbb{F}^n,\ (\mathbb{F}^n)^*,\ \Lambda^2 \mathbb{F}^n,\ (\Lambda^2 \mathbb{F}^n)^*.$$

Moreover, all r-discontinuous fundamental representations of SL_n are 2-discontinuous.

Proof. The closed G-orbit $\mathbb{X} \subset \mathbb{P}(\Lambda^k \mathbb{F}^n)$ is the Grassmann variety $\mathrm{Gr}_k(\mathbb{F}^n)$ under its Plücker embedding. It is well known that a suitable isomorphism between $\Lambda^k \mathbb{F}^n$ and $\Lambda^{n-k} \mathbb{F}^n$ induces an isomorphism between the respective projective spaces, which carries $\mathrm{Gr}_k(\mathbb{F}^n)$ to $\mathrm{Gr}_{n-k}(\mathbb{F}^n)$. Hence, for our purposes, it is sufficient to consider $k \leq n/2$.

The fact that $V(\pi_1) = \mathbb{F}^n$ and $V(\pi_2) = \Lambda^2 \mathbb{F}^n$ are rs-continuous is well known. In fact, in the first case we have $\mathbb{X} = \mathbb{P}(\mathbb{F}^n)$, so all vectors have rank 1. In the second case, $\Lambda^2 \mathbb{F}^n$ can be identified with the space of skew-symmetric $n \times n$ matrices. Such a matrix has even rank (in the usual sense) and the SL_n-orbit X through a highest weight vector in $\Lambda^2 \mathbb{F}^n$ consists of all matrices of rank 2. A skew-symmetric matrix ψ of rank $2r$ can be written as a sum of r skew-symmetric matrices of rank 2, and so $\mathrm{rk}_{\mathbb{X}}[\psi] = r$. We can now see that the set

$$\{[\psi] \in \mathbb{P}(\Lambda^2 \mathbb{F}^n) : \mathrm{rk}_{\mathbb{X}} \psi \leq r\}$$

is closed for every r. This completes the argument in this case.

We now turn to the remaining cases. Due to the duality $\Lambda^k \mathbb{F}^n \leftrightarrow (\Lambda^{n-k} \mathbb{F}^n)^*$, it suffices to consider $n \geq 6$. Proposition 5.2 implies that, to show that $\Lambda^k \mathbb{F}^n$ ($3 \leq k \leq n/2$) is 2-discontinuous, it is sufficient to show that $\Lambda^3 \mathbb{F}^6$ is 2-discontinuous.

Lemma 6.3. *The representation of SL_6 on $\Lambda^3 \mathbb{F}^6$ is 2-discontinuous.*

Proof. It is shown in [32, Ch. III, Thm 1.4], that $\sigma_2(X(\Lambda^3\mathbb{F}^6)) = \Lambda^3\mathbb{F}^6$ and therefore it is enough to show that $\Lambda^3\mathbb{F}^6$ contains a vector of rank 3 or more. To do this we count the number of orbits of vectors of rank 0, 1, 2 and compare this number with the known number of orbits for the action of SL_6 on $\Lambda^3\mathbb{F}^6$, see [16] or [28].

By definition there is one orbit of vectors of rank 0 and one orbit of vectors of rank 1. Let us consider the vectors of rank 2 in V. We shall show that there are two orbits of such vectors. Any vector of rank 2 can be written as

$$\psi = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6,$$

with some $v_j \in V$. The first possibility is that $v_1, \ldots, v_6$ form a basis of $\mathbb{F}^6$. This is indeed the generic situation. If suitable Borel and Cartan subgroups of SL_6 are chosen, the two summands of ψ are, respectively, the highest and lowest weight vectors in V. The group GL_6 acts transitively on the set of all bases of $\mathbb{F}^6$; the group SL_6 acts transitively on the set of their projective images. Thus the points of the first type form a single G-orbit $\mathbb{X}_2'$ which is open in $\mathbb{P}(\Lambda^3\mathbb{F}^6)$. We denote by Z the complement to this orbit in $\mathbb{P}(\Lambda^3\mathbb{F}^6)$. The second possibility is to have

$$\dim(\langle v_1, v_2, v_3\rangle \cap \langle v_4, v_5, v_6\rangle) = 1.$$

If this is the case, by changing the vectors if necessary, we may reduce to the situation where $v_1 = v_4$ and

$$\psi = v_1 \wedge (v_2 \wedge v_3 + v_5 \wedge v_6), \quad \text{with} \quad \langle v_2, v_3\rangle \cap \langle v_5, v_6\rangle = 0.$$

Since $v_2 \wedge v_3 + v_5 \wedge v_6$ has rank 2 in $\Lambda^2\mathbb{F}^6$ (with respect to $\mathrm{Gr}_2(\mathbb{F}^6)$), we deduce that ψ has indeed rank 2 in V. The point ϕ does not belong to $\mathbb{X}_2'$, because the action of GL_6 respects linear dependencies. On the other hand, it is also clear that GL_6 acts transitively on the set $\mathbb{X}_2''$ of points of this second type, and hence SL_6 acts transitively on the set of their images in $\mathbb{P}$. Note that

$$\text{if } \dim(\langle v_1, v_2, v_3\rangle \cap \langle v_4, v_5, v_6\rangle) > 1, \text{ then } \mathrm{rk}[\psi] = 1.$$

We can conclude that there are exactly two G-orbits consisting of points of rank 2, namely

$$\mathbb{X}_2' = \mathbb{P}\backslash Z, \quad \mathbb{X}_2'' = Z \cap \mathbb{X}_2.$$

Thus there are four SL_6-orbits of vectors of rank 0, 1, 2. It is known that SL_6 has five orbits in $\Lambda^3\mathbb{F}^6$. Therefore $\Lambda^3\mathbb{F}^6$ has a unique SL_6-orbit of vectors of rank 3 or more and $\Lambda^3\mathbb{F}^6$ is 2-discontinuous. An example of a vector of rank 3 is (see [16])

$$\Lambda 3 = v_1 \wedge v_2 \wedge v_4 + v_1 \wedge v_5 \wedge v_3 + v_6 \wedge v_2 \wedge v_3.$$

□

6.2. $G = SO_n$

Let $\ell = rank(G) = \lfloor \frac{n}{2} \rfloor$. In this section we prove the following proposition.

Proposition 6.4. *The natural representation $V(\pi_1)$ is rs-continuous.*

(1) If n is even, then, for $j = 2, \dots, \ell - 2$, the representation $V(\pi_j)$ is r-discontinuous.
(2) If n is odd, then, for $j = 2, \dots, \ell - 1$, the representation $V(\pi_j)$ is r-discontinuous.
(3) The spin representations ($V(\pi_{\ell-1})$ and $V(\pi_\ell)$ for even n and $V(\pi_\ell)$ for odd n) are rs-continuous if and only if $n \leq 10$.

Moreover, all fundamental representations of SO_n which are r-discontinuous are 2-discontinuous.

Proof. The first statement is well known. Indeed, the group SO_n has exactly two orbits in $\mathbb{P}(\pi_1)$, namely, the quadric and its complement. The first one consists, by definition, of points of rank 1. The second one consists necessarily of points of rank 2.

The second and third statement in the proposition are concerned with fundamental representations of SO_n, which are not the natural nor the spin representation. We handle the two statements at once. In the case $j = 2$, the representation is actually the adjoint representation, i.e. $V(\pi_2) = \mathfrak{so}_n$. Here results of [20] show that $\sigma_2(\mathbb{X}) \neq \mathbb{X} \sqcup \mathbb{X}_2$. Thus the representation is 2-discontinuous. The remaining cases, $j \geq 3$, are reduced to the case $j = 2$ via Proposition 5.2.

Now, we turn to the last statement of the proposition, concerning the spin representations. First, recall that, for even n, the geometric properties we are concerned with are the same for the two spin representations $V(\pi_{\ell-1})$ and $V(\pi_\ell)$. Also, either one of these representations remains irreducible when restricted to $Spin_{n-1}$ and, furthermore, $Spin_{n-1}$ acts transitively on the closed orbit of $Spin_n$ in $\mathbb{P}(\pi_\ell)$. Thus, it is enough to check statement 3) for the representations $V(\pi_\ell)$ of $Spin_{2\ell}$. Let $\mathbb{X}$ denote the closed orbit of $Spin_{2\ell}$ in $\mathbb{P}(\pi_\ell)$.

It is shown, in [8, Section 3.5], that for $2\ell = 12$ the secant variety $\sigma_2(\mathbb{X})$ contains elements of rank 3. Thus the representation $V(\pi_6)$ of $Spin_{12}$ is 2-discontinuous. Using Proposition 5.1, we deduce that the representation $V(\pi_\ell)$ of $Spin_{2\ell}$ is r-discontinuous for all $\ell \geq 6$. So, according to the remarks made earlier in this proof, the spin representations of $Spin_n$ are 2-discontinuous for $n \geq 11$.

It remains to verify that the spin representations are rs-continuous for even $n \leq 10$. This statement easily follows from Corollary 2.3. This completes the proof of the proposition. □

6.3. $G = Sp_{2n}$

Proposition 6.5. *The fundamental representations of Sp_{2n} which are rs-continuous are exactly $V(\pi_1)$ and $V(\pi_2)$. All other fundamental representations of Sp_{2n} are 2-discontinuous.*

Proof. The representation $V(\pi_1)$ is simply the natural representation of Sp_{2n} on $\mathbb{F}^{2n}$. The action of Sp_{2n} on $\mathbb{P}(\mathbb{F}^{2n})$ is transitive, i.e. $\mathbb{X} = \mathbb{P}(\mathbb{F}^{2n})$ and there is nothing more to prove here. The representations $V(\pi_2)$ and $V(\pi_k)$, $k \geq 3$ are considered in Lemmas 6.8 and 6.7, respectively. □

Lemma 6.6. *The representation $V(\pi_3)$ of Sp_{2n} is 2-discontinuous for $n \geq 3$.*

Proof. Let $n \geq 3$ and consider Sp_{2n} to be defined with respect to the skew-symmetric form

$$(z_1 \wedge z_6 + z_2 \wedge z_5 + z_3 \wedge z_4) + (z_7 \wedge z_8 + \ldots + z_{2n-1} \wedge z_{2n})$$

on $\mathbb{F}^{2n}$, where $z_1, \ldots, z_{2n}$ is the dual basis corresponding to the basis $v_1, \ldots, v_{2n}$ of $\mathbb{F}^{2n}$.

Let $\mathbb{X}$ be the set of points $[w_1 \wedge w_2 \wedge w_3]$ such that $w_1, w_2, w_3 \in \mathbb{F}^{2n}$ span a 3-dimensional isotropic subspace $\mathbb{F}^{2n}$. Let X be the affine cone over $\mathbb{X}$. We set

$$\Lambda_0^3 \mathbb{F}^{2n} := \langle X \rangle.$$

The set $\mathbb{X}$ is a single Sp_{2n}-orbit and thus $\Lambda_0^3 \mathbb{F}^{2n}$ is an irreducible Sp_{2n}-module. There is a unique up to scaling Sp_{2n}-isomorphism between $V(\pi_3)$ of Sp_{2n} and $\Lambda_0^3 \mathbb{F}^{2n}$. We identify $V(\pi_3)$ with $\Lambda_0^3 \mathbb{F}^{2n}$. The varieties $\mathbb{X}$ and $\mathbb{X}(\pi_3)$ coincide under this identification.

Note that if $\psi \in V(\pi_3)$ has rank 3 as a vector of the SL_{2n}-module $\Lambda^3 \mathbb{F}^{2n}$, then ψ has rank 3 or more as a vector of the Sp_{2n}-module $V(\pi_3)$.

Consider the tensor $\Lambda 3 \in \Lambda^3 \mathbb{F}^6$ given at the end of the proof of Lemma 6.3. One has

$$[v_1 \wedge v_2 \wedge v_4], [v_1 \wedge v_5 \wedge v_3], [v_6 \wedge v_2 \wedge v_3] \in \mathbb{X}(\pi_3)$$

and thus $\Lambda 3 \in \Lambda_0^3 \mathbb{F}^6 \subset \Lambda_0^3 \mathbb{F}^{2n}$. Moreover the rank of $\Lambda 3$ is 3 or less. As $\Lambda 3$ has rank 3 as an element of SL_6-module $\Lambda^3 \mathbb{F}^6$, $\Lambda 3$ has rank 3 as an element of $\Lambda^3 \mathbb{F}^{2n}$ (see Proposition 5.2). Hence

$$\mathrm{rk}_{\mathbb{X}}\, \Lambda 3 = 3, \tag{5}$$

where $\Lambda 3$ is considered as an element of $\Lambda_0^3 \mathbb{F}^{2n}$.

It is shown in [32, Ch. III, Thm 1.4], that $\sigma_2(\mathbb{X}) = \mathbb{P}(\Lambda_0^3 \mathbb{F}^6)$. Thus $\Lambda 3$ has border rank 2 or less as an element of $\Lambda_0^3 \mathbb{F}^6$ and hence

$$\underline{\mathrm{rk}}_{\mathbb{X}} \Lambda 3 \leq 2,$$

here $\Lambda 3$ is considered as an element of $\Lambda_0^3 \mathbb{F}^{2n}$. Therefore the Sp_{2n}-module $V(\pi_3) = \Lambda_0^3 \mathbb{F}^{2n}$ is 2-discontinuous.

Lemma 6.7. *Fix $n \geq k \geq 3$. The representation $V(\pi_k)$ of Sp_{2n} is 2-discontinuous.*

Proof. If $n \geq k \geq 3$, the Dynkin diagram C_n of Sp_{2n} has a unique subdiagram $\underline{C}_{n,k}$ of type C_{n-k+3}. The chopping of π_k to this diagram equals π_3. By Lemma 6.6, $V(\pi_3)$ is not rs-continuous for $\underline{G} = Sp_{2(n-k+3)}$ (this group corresponds to the

Dynkin diagram $\underline{C}_{n,k}$) and thus $V(\pi_k)$ is not an rs-continuous Sp_{2n}-module by Proposition 5.1. □

We are now going to prove that $V(\pi_2)$ is rs-continuous for $Sp_{2n}(n \geq 2)$. To do this we set $V := \mathbb{F}^{2n}$ and fix a nondegenerate antisymmetric bilinear form ω on V. Note that the second fundamental module of $\mathfrak{sp}(V)$ is isomorphic to the set of vectors in $\Lambda^2 V$, which are annihilated by ω (here we consider ω as an element of $(\Lambda^2 V)^*$). We denote this space by $\Lambda_0^2 V$. To complete the proof of Proposition 6.5 we prove the following lemma.

Lemma 6.8. *(a) For any $\underline{\omega} \in \Lambda_0^2 V$ the rank of $\underline{\omega}$ as a bilinear coform is twice the rank of $\underline{\omega}$ as a vector in an $Sp(V)$-module.*
(b) The $Sp(V)$-module $\Lambda_0^2 V$ is rs-continuous.

To prove Lemma 6.8 we introduce a notion related to bilinear coforms $\underline{\omega} \in \Lambda^2 V$. A bilinear coform $\underline{\omega}$ defines a map $V^* \to V$ by $v \to \underline{\omega}(v, \cdot)$. We denote the image of this map by $\operatorname{Supp} \underline{\omega}$. We have natural inclusions

$$\Lambda^2 \operatorname{Supp} \underline{\omega} \to \Lambda^2 V, \quad \Lambda_0^2 \operatorname{Supp} \underline{\omega} \to \Lambda_0^2 V,$$

and if $\underline{\omega} \in \Lambda_0^2 V$, then $\underline{\omega} \in \Lambda_0^2 \operatorname{Supp} \underline{\omega}$. Note that $\underline{\omega}$ is nondegenerate as an element of $\Lambda^2 \operatorname{Supp} \underline{\omega}$ and, in particular, defines a bilinear form $\underline{\omega}^*$ on $\operatorname{Supp} \underline{\omega}$ (there is no canonical way to extend $\underline{\omega}^*$ to the whole V).

Lemma 6.8 follows from Lemma 6.9 below; a proof of Lemma 6.8 is presented after the proof of Lemma 6.9.

Lemma 6.9. *Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. Then there exist a set of elements $x_1, \ldots, x_r, y_1, \ldots, y_r \in \operatorname{Supp} \underline{\omega}$ such that*

$$\underline{\omega} = x_1 \wedge y_1 + \cdots + x_r \wedge y_r, \quad \textit{and} \quad \omega(x_i, y_i) = 0 \ \ \textit{for all} \ \ i.$$

In turn, Lemma 6.9 follows from Lemma 6.10 below; a proof of Lemma 6.9 is presented after the proof of Lemma 6.10.

Lemma 6.10. *Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. If $r > 0$, then there exist elements $x_1, y_1 \in \operatorname{Supp} \underline{\omega}$ such that*

$$\operatorname{rank}(\underline{\omega} - x_1 \wedge y_1) = 2r - 2, \quad \textit{and} \quad \omega(x_1, y_1) = 0,$$

where rank(η) denotes the usual rank of a bilinear coform η.

In turn, Lemma 6.10 follows from Lemma 6.11 below; a proof of Lemma 6.10 is presented after the proof of Lemma 6.11.

Lemma 6.11. *Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. If $r > 0$, then there exists an open subset $U \subset \operatorname{Supp} \underline{\omega}$ such that for any $x_1 \in U$ there exists $y_1 \in \operatorname{Supp} \underline{\omega}$ such that $\underline{\omega}^*(x_1, y_1) \neq 0$ and $\omega(x_1, y_1) = 0$.*

Proof. If a form ω is zero on $\operatorname{Supp}\underline{\omega}$, then for any non-zero $x_1 \in \operatorname{Supp}\underline{\omega}$ there exists $y_1 \in \operatorname{Supp}\underline{\omega}$ such that $\underline{\omega}^*(x_1, y_1) \neq 0$, because the form $\underline{\omega}^*$ is nondegenerate on $\operatorname{Supp}\underline{\omega}$. In this case $\omega(x_1, y_1) = 0$, because $\omega = 0$.

We assume that ω is non-zero on $\operatorname{Supp}\underline{\omega}$. Since $\underline{\omega} \in \Lambda_0^2 V$, the pairing of $\underline{\omega}$ with ω equals 0. Thus $[\underline{\omega}^*] \neq [\omega|_{\Lambda^2 \operatorname{Supp}\underline{\omega}}]$. Hence, for some open subset $U \subset \operatorname{Supp}\underline{\omega}$ and any $x_1 \in U$, both $\omega(x_1, \cdot)$, $\underline{\omega}^*(x_1, \cdot)$ are non-zero and

$$[\omega(x_1, \cdot)] \neq [\underline{\omega}^*(x_1, \cdot)].$$

Therefore for any $x_1 \in U$ there exists $y_1 \in \operatorname{Supp}\underline{\omega}$ such that $\underline{\omega}^*(x_1, y_1) \neq 0$ and $\omega(x_1, y_1) = 0$. □

Proof of Lemma 6.10. Let (x_1, y_1) be a pair as in Lemma 6.11. We denote by W_2 the space spanned by x_1, y_1 and by W_{2r-2} the orthogonal complement to W_2 in $\operatorname{Supp}\underline{\omega}$ with respect to $\underline{\omega}$. Thanks to the choice of x_1, y_1, the form $\underline{\omega}^*$ is nondegenerate on W_2 and therefore $\operatorname{Supp}\underline{\omega} = W_2 \oplus W_{2r-2}$. Then $\underline{\omega} = \underline{\omega}^2 + \underline{\omega}^{2r-2}$ for uniquely determined coforms $\underline{\omega}^2 \in \Lambda^2 W_2$ and $\underline{\omega}^{2r-2} \in \Lambda^2 W_{2r-2}$. We have $\underline{\omega}^2 = \lambda(x_1 \wedge y_1) = x_1 \wedge (\lambda y_1)$ for some $\lambda \in \mathbb{F}^\times$. Therefore

$$\operatorname{rk}(\underline{\omega} - x_1 \wedge (\lambda y_1)) = 2r - 2,$$

and $\omega(x_1, (\lambda y_1)) = 0$. □

Proof of Lemma 6.9. To prove Lemma 6.9 we use induction.

The r-th statement of the induction is: Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. Then there exist a set of elements $x_1, \ldots, x_r, y_1, \ldots, y_r \in \operatorname{Supp}\underline{\omega}$ such that

$$\underline{\omega} = x_1 \wedge y_1 + \ldots + x_r \wedge y_r, \quad \text{and} \quad \omega(x_i, y_i) = 0 \quad \text{for all} \quad i.$$

Basis of the induction, for $r = 1$: Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank 2. Then there exist elements $x_1, y_1 \in \operatorname{Supp}\underline{\omega}$ such that $\underline{\omega} = x_1 \wedge y_1$, and $\omega(x_1, y_1) = 0$.

First, we check the basis of the induction. Let x_1, y_1 be basis of $\operatorname{Supp}\underline{\omega}$. Then $\underline{\omega} = \lambda x_1 \wedge y_1$ for some $\lambda \in \mathbb{F}^\times$. As $\underline{\omega} \in \Lambda_0^2 V$, we have $\omega(\underline{\omega}) = \omega(\lambda x_1 \wedge y_1) = \omega(x_1, \lambda y_1) = 0$. Then $\underline{\omega} = x_1 \wedge (\lambda y_1)$ and $\omega(x_1, \lambda y_1)$=0.

Now we prove that the r-th statement of the induction follows from the $(r-1)$-th statement. We assume that the $(r-1)$-th statement holds. According to Lemma 6.10 there exists $x_r, y_r \in \operatorname{Supp}\underline{\omega}$ such that

$$\operatorname{rk}(\underline{\omega} - x_r \wedge y_r) = 2r - 2, \quad \text{and} \quad \omega(x_r, y_r) = 0.$$

Note that $\omega(x_r \wedge y_r) = \omega(x_r, y_r) = 0$ and therefore $\underline{\omega} - x_r \wedge y_r \in \Lambda_0^2 V$. By hypothesis, there exist $x_1, \ldots, x_{r-1}, y_1, \ldots, y_{r-1} \in \operatorname{Supp}(\underline{\omega} - x_r \wedge y_r) \subset \operatorname{Supp}\underline{\omega}$ such that

$$\underline{\omega} - x_r \wedge y_r = x_1 \wedge y_1 + \cdots + x_{r-1} \wedge y_{r-1}, \quad \text{and} \quad \omega(x_i, y_i) = 0 \quad \text{for all} \quad i.$$

This completes the proof of Lemma 6.9. □

Proof of Lemma 6.8. First note that a highest weight vector of the $Sp(V)$-module $\Lambda_0^2 V$ is a wedge product of two ω-orthogonal vectors of V. Fix a coform $\underline{\omega} \in \Lambda_0^2 V$. A sum of r vectors from the $Sp(V)$-orbit of a highest weight vector has rank at most $2r$ as a bilinear coform. Hence the rank of $\underline{\omega}$ as a vector of an $Sp(V)$-module is not less than half the rank of $\underline{\omega}$ as a bilinear coform. On the other hand, Lemma 6.9 implies that the rank of $\underline{\omega}$ as a vector of an $Sp(V)$-module is not larger than half the rank of $\underline{\omega}$ as a bilinear coform. Therefore the rank of $\underline{\omega}$ as a vector of an $Sp(V)$-module is equal to half the rank of $\underline{\omega}$ as a bilinear coform. This proves part a) of Proposition 6.8.

The set of coforms of rank r or less is closed for all r. This completes the proof of part b). □

7. Fundamental representations (exceptional groups)

In this section we prove Theorem 1.1 for fundamental modules of exceptional groups, i.e. we prove Theorem 7.1. Essentially, we consider case-by-case all 27 fundamental representations of the 5 exceptional groups and provide some arguments for each case, by which the corresponding fundamental module is r-discontinuous or rs-continuous. The result is presented below.

Theorem 7.1. *Assume that $V(\lambda)$ is a fundamental effective G-module. Then $V(\lambda)$ is rs-continuous if and only if the pair $(G, V(\lambda))$ appears in the following table.*

G	*Representation V*	*Highest weight of V*
E_6	$\mathbb{F}^{27}$, $(\mathbb{F}^{27})^*$	π_1, π_5
F_4	$\mathbb{F}^{26}$	π_1
G_2	$\mathbb{F}^7$	π_1

, (6)

where the notation is the same as in Theorem 1.1.

Moreover, all r-discontinuous fundamental representations of exceptional groups are 2-discontinuous.

The types of arguments are presented in the following tables.

Symbol	Argument for being **rs-continuous**	References
SM	The representation is reduced to a subminuscule representation	Section 1, [4]
F4C	The representation is equivalent to $V(\pi_1)$ of F_4	Prop. 7.2, Sect. 7.1

Symbol	Argument for being **r-discontinuous**	References
CC	The representation is chopable to an r-discontinuous representation of some classical group	—
Ad	The representation is adjoint	Section 1, [19]
AdC	The representation is chopable to the adjoint representation of some exceptional group	—
F4D	The representation is equivalent to $V(\pi_2)$ of F_4	Prop. 7.9, Sect. 7.2
E7D	The representation is equivalent to the E_7-representation $\mathrm{V}(\pi_1)$	Prop. 7.13, Sect. 7.3

In the following tables, we provide, for each fundamental representation of each exceptional group, an argument by which it is rs-continuous or r-discontinuous.

F. weights of E_6	π_1	π_2	π_3	π_4	π_5	π_6
Arguments	SM	CC	CC	CC	SM	CC or Ad

F. weights of E_7	π_1	π_2	π_3	π_4	π_5	π_6	π_7
Arguments	E7D	CC	CC	CC	CC	Ad	CC

F. weights of E_8	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
Arguments	Ad	CC	CC	CC	CC	CC	AdC	CC

F. weights of F_4	π_1	π_2	π_3	π_4
Arguments	F4C	F4D	CC	Ad

F. weights of G_2	π_1	π_2
Arguments	SM	Ad

For the representations $V(\pi_k)$ of exceptional groups $E_n (n = 6, 7, 8)$, for which argument CC is applicable, chopping of V in the vertex with number $n-1$ is a 2-discontinuous representation of a classical group of type D_{n-1}. To apply argument AdC one should chop vertex with number 1. For all representations of the exceptional group F_4, to apply argument CC one can always chop the vertex with number 1.

Let us first justify arguments SM, CC, Ad, AdC.

SM) It is shown in [4] that any subminuscule representation is rs-continuous, i.e. that rank and border rank coincide for such representations.

CC) According to Proposition 5.1, if some chopping $\underline{V}$ of a representation V is 2-discontinuous, then V is 2-discontinuous.

Ad) According to [19], all adjoint representations of exceptional groups are 2-discontinuous.

AdC) According to Proposition 5.1 and Ad), if some chopping $\underline{V}$ of a representation V is an adjoint representation of an exceptional group, then V is 2-discontinuous.

The rest of this section is devoted to the justification of arguments F4C, F4D and E7W.

7.1. Rs-continuity of $V(\pi_1)$ for F_4

In this subsection we prove the following.

Proposition 7.2. *The fundamental representation $V(\pi_1)$ of F_4 is rs-continuous.*

Proof. It is known, [32, p. 59], that the generic rank of $V(\pi_1)$ is three, so that

$$\sigma_3(\mathbb{X}(\pi_1)) = \mathbb{P}(V(\pi_1)).$$

In Lemmas 7.3 and 7.4 below, we show that $V(\pi_1)$ is 2- and 3-continuous, respectively, which implies that this module is rs-continuous. □

Lemma 7.3. *The F_4-module $V(\pi_1)$ is 2-continuous.*

Lemma 7.4. *The F_4-module $V(\pi_1)$ is 3-continuous.*

The first fundamental representation $V(\pi_1)$ of F_4 is 26-dimensional and is the (nontrivial) representation of the smallest possible dimension for this group. The discussion which follows involves several representations of various groups. This would make the notation $V(\lambda)$ ambiguous. We have chosen to denote the representations spaces by indices corresponding to their dimension. The set of highest weight vectors, previously denoted by $X(\lambda)$, will be denoted by $X(V)$. We let V_{26} denote the representation space of $(F_4, V(\pi_1))$ and $X(V_{26})$ be the set of highest weight vectors. To study V_{26} we use the fact that it can be obtained as a generic hyperplane in the smallest, 27-dimensional representation of E_6, which we denote by $V_{27} = (E_6, V(\pi_1))$. We summarize some known results in the following lemma.

Lemma 7.5. (i) *The algebra of E_6-invariant polynomials on V_{27} is polynomial in one generator of degree 3, i.e. $\mathbb{F}[V_{27}]^{E_6} = \mathbb{F}[DET]$, where $DET \in S^3(V_{27}^*)$.*

(ii) *The orbits of E_6 in V_{27} are the following:*

$$0,\ X(V_{27}),\ \{DET = 0\}\backslash\overline{X(V_{27})},\ \{DET = a\},\ a \in \mathbb{F}^\times;$$

their dimensions are, respectively, 0, 17, 26, 26. *The orbits of $E_6 \times \mathbb{F}^\times$ in V_{27} are the following (lower indices indicate dimension):*

$$\mathcal{O}_0 = \{0\},\quad \mathcal{O}_{17} = X(V_{27}),\quad \mathcal{O}_{26} = \{DET = 0\}\backslash\overline{X(V_{27})},\quad \mathcal{O}_{27} = \{DET \neq 0\}.$$

(iii) *There are exactly three E_6 orbits in the projective space $\mathbb{P}(V_{27})$ and they are exactly the rank subsets with respect to $\mathbb{X}(V_{27})$, namely,*

$$\mathbb{X}(V_{27}),\quad \mathbb{X}_2(V_{27}) = \{DET = 0\}\backslash\mathbb{X}(V_{27}),\quad \mathbb{X}_3(V_{27}) = \{DET \neq 0\};$$

their dimensions are, respectively, 16, 25, 26. *The secant varieties of $\mathbb{X}(V_{27})$ are exactly the closures of the E_6-orbits in $\mathbb{P}(V_{27})$.*

(iv) *The stabilizer of any vector $v \in \{DET \neq 0\}$ is isomorphic to F_4. The orth-complement $v^\perp \subset V_{27}$ is an irreducible F_4-module isomorphic to V_{26}, i.e. $V_{27} \cong \langle v\rangle \oplus V_{26}$ as F_4-modules.*

(v) *The secant varieties of* $\mathbb{X}(V_{26})$ *are obtained as intersections of the secant varieties of* $\mathbb{X}(V_{27})$ *with the hyperplane* $\mathbb{P}(V_{26})$*, i.e.*

$$\mathbb{X}(V_{26}) = \mathbb{P}(V_{26}) \cap \mathbb{X}(V_{27}), \quad \sigma_2(\mathbb{X}(V_{26})) = \mathbb{P}(V_{26}) \cap \sigma_2(\mathbb{X}(V_{27})),$$
$$\sigma_3(\mathbb{X}(V_{26})) = \mathbb{P}(V_{26}).$$

Proof. Since the results are known, but are a compilation of the work of many authors, we confine ourselves to giving references (not necessarily the original ones) for the various parts of the lemma. Part (i) can be found in Table II in [18]. As for part (ii), the fact that

$$\{DET = a\}, a \neq 0$$

is a single E_6-orbit, is proven in [18, Proposition 1.1], while the enumeration of the orbits in the nullcone $\{DET = 0\}$ is given in [32, p. 59]. Part (iii) can also be deduced from the discussion on p. 59 of [32] or can be seen to follow directly from the fact that V_{27} is a subcominuscule representation and for such representations the rank sets are exactly the group orbits in the projective space, cf. [4, 4]. Parts (iv) and (v) are also quoted from [32, p. 59-60]. □

The above proposition and, specifically, parts (iv) and (v) allow us to practically forget about the group F_4 and use only properties of V_{27} and a generic hyperplane inside it. We shall need to understand the structure of V_{27} with respect to a subgroup of E_6 of type D_5. Let $H \subset E_6$ be the regular subgroup whose root system is generated by the set of simple roots $S := \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ (we use the numbering of simple roots as in [27, p. 292]). It turns our that $H \cong Spin_{10}$.

Lemma 7.6. *The decomposition, as an* H*-module, of the simple* E_6*-module* V_{27} *is*

$$V_{27} \cong V_1 \oplus RSpin_{10} \oplus V_{10},$$

where V_1 *is a one-dimensional trivial* H*-module,* $RSpin_{10}$ *is the spinor* H*-module, and* V_{10} *is the natural representation of* SO_{10} *(recall that* $Spin_{10}$ *is a cover of* SO_{10}*).*

Proof. The result is obtained by a straightforward consideration of the weights of the modules involved, using the fact that H is a regular subgroup of E_6. □

Lemma 7.7. *The nonzero isotropic vectors in* V_{10} *belong to* $X(V_{27})$ *and have rank 1 as elements of the* E_6*-module* V_{27}*. The non-isotropic vectors in* V_{10} *belong to* $\mathcal{O}_{26}$ *and have rank 2 as elements of the* E_6*-module* V_{27}*.*

Proof. Since H is a regular subgroup of E_6, the weight spaces for E_6 in V_{27} are also weight spaces for H. Thus V_{10} is a span of some of these weight spaces. The E_6-weights of V_{27} are

$$\varepsilon_i \pm \varepsilon, -\varepsilon_i - \varepsilon_j \ (i \neq j).$$

The weights appearing in V_{10} are

$$\varepsilon_i - \varepsilon, -\varepsilon_1 - \varepsilon_i \ (i \neq 1).$$

The weight $-\varepsilon_6 - \varepsilon$ is the lowest weight of V_{27} and thus any element of the corresponding weight space belongs to $\mathcal{O}_{17}$. On the other hand, any element of the weight space of weight $-\varepsilon_6 - \varepsilon$ is isotropic. Since all isotropic vectors of V_{10} are conjugate by SO_{10}, all isotropic vectors of V_{10} belong to $\mathcal{O}_{17}$.

It remains to show that all non-isotropic vectors of V_{10} (they are all $SO_{10} \times \mathbb{F}^\times$-conjugate) belong to $\mathcal{O}_{26}$. To this end, we note that the weights of V_{10}

$$\varepsilon_2 - \varepsilon, -\varepsilon_1 - \varepsilon_2$$

are, respectively, the highest and the lowest weight of V_{27} with respect to the set of simple roots of E_6

$$\Pi' = \{\varepsilon_2 - \varepsilon_1, \varepsilon_1 + \varepsilon_4 + \varepsilon_5 - \varepsilon, \varepsilon_6 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, -\varepsilon_4 - \varepsilon_3 - \varepsilon_6 + \varepsilon, \varepsilon_3 - \varepsilon_6\}.$$

Thus $v^{\varepsilon_2 - \varepsilon} + v^{-\varepsilon_1 - \varepsilon_2} \in \mathcal{O}_{26}$ [32, Ch. III, Thm 1.4]. Since all non-isotropic vectors of V_{10} are $SO_{10} \times \mathbb{F}^\times$-conjugate, all non-isotropic vectors of V_{10} belong to $\mathcal{O}_{26}$. □

Proof of Lemma 7.3. According to Lemma 7.5, to prove that V_{26} is 2-continuous it suffices to show that for any $x \in V_{26} \cap \mathcal{O}_{26}$ there exist $x_+, x_- \in V_{26} \cap \mathcal{O}_{17}$ such that $x = x_+ + x_-$. We fix $x \in V_{26} \cap \mathcal{O}_{26}$. First, we note that there exists a Borel subalgebra $\mathfrak{b} \subset E_6$ with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ such that $x \in V_{10}$ (we use the notation of Lemma 7.7). Then $x \in V_{10} \cap \mathcal{O}_{26}$ and thus x is a non-isotropic vector of V_{10} by Lemma 7.7. Let $x^\perp$ be the orthogonal complement to x in V_{10}. Note that a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ of V_{10} is still nondegenerate after restriction to $x^\perp$.

Since V_{26} is a 26 dimensional subspace of the 27 dimensional space V_{27}, we have

$$\dim(V_{26} \cap x^\perp) \geq \dim(x^\perp) - 1 = 9.$$

As $9 > \frac{1}{2} \dim V_{10}$, the restriction of $(\cdot, \cdot)$ to $x^\perp \cap V_{26}$ is non-zero. Hence there exists $y \in x^\perp \cap V_{26}$ such that $(y, y) = \frac{-(x,x)}{4}$. Set

$$x_+ = \frac{x}{2} + y, \quad x_- = \frac{x}{2} - y.$$

We have

$$(x_+, x_+) = \frac{1}{4}(x, x) + (y, y) = 0 = (x_-, x_-) \quad \text{and} \quad x_+ + x_- = x,$$

i.e. the vectors $x_\pm$ are isotropic and their sum is equal to x. Thus $x_\pm \in \mathcal{O}_{17}$. Since $y \in V_{26}$, we have $x_\pm \in V_{26}$. Hence $x_\pm \in X(V_{26})$. Therefore V_{26} is 2-continuous. □

Before proceeding with the proof of Lemma 7.4, we need the following auxiliary result.

Lemma 7.8. *Let V be a finite-dimensional vector space. Let Z_f be a hypersurface determined as the zero-locus of a non-zero homogeneous polynomial $f \in \mathbb{F}[V]$ and let $X \in V$ be a conical subset spanning V. Then, $V = Z_f + X$, i.e. for every $v \in V$ there exist $v_2 \in Z_f$ and $x \in X$ such that $v = v_2 + x$.*

Proof. Assume on the contrary, that there exists $v \in V$ such that $v \neq v_2 + x$ for any $v_2 \in Z_f$ and $x \in X$. Then $f(v + tx) \neq 0$ for any $x \in X$ and any $t \in \mathbb{F}$. The function $f(v+tx)$ is polynomial and thus $f(v+tx) \neq 0$ for any $t \in \mathbb{F}$ if and only if $f(v + tx)$ is a non-zero constant as a polynomial of t. Hence the first derivative of $f(v+tx)$ with respect to t is zero for $t = 0$, i.e. the value of df in the direction x at the point v is zero. As X spans V, $df = 0$ at v.

We claim that $f(w) = 0$ for all points $w \in V$ such that $df = 0$ at w (equivalent to: all partial derivatives of f vanish at w). Indeed, the set of equations $df = 0$ determines some subvariety Z_{defn} of V and it suffices to show that $f = 0$ at any smooth point of any irreducible component of Z_{defn}. Obviously $df = 0$ on the smooth locus of any irreducible component of Z_{defn}. Thus f is constant on any irreducible component of Z_{defn}. Since f is homogeneous, $f(0) = 0$, and any irreducible component of Z_{defn} contains 0. Thus $f|_{Z_{defn}} = 0$.

Compiling the previous two paragraphs, we obtain $f(v) = 0$. Thus $v = v + 0$, where $v \in Z_f$ and $0 \in X$. This completes the proof. $\square$

Proof of Lemma 7.4. The secant variety $\sigma_2(X(V_{26}))$ is the zero-locus of some homogeneous function DET of degree 3 and $X(\pi_1)$ spans V_{26}. Thus, according to Lemma 7.8, any vector

$$x \in V_{26} \backslash \sigma_2(X(V_{26}))$$

may be represented as $x = x_1 + x_2$, for some $x_1 \in X(\pi_1)$ and $x_2 \in \sigma_2(X(V_{26}))$. Therefore, by Lemma 7.3, any vector in $V_{26} \backslash \sigma_2(X(V_{26}))$ has rank 3. $\square$

7.2. R-discontinuity of $V(\pi_2)$ for F_4

The goal of this section is to prove the following.

Proposition 7.9. *The fundamental representation $V(\pi_2)$ of F_4 is 2-discontinuous.*

We deduce this proposition from the following three lemmas (we use the description of the corresponding roots and weights given in [27, p. 294–295]). In particular, the highest root of $\mathfrak{f}_4$ coincides with the fundamental weight π_4. In the rest of this section we use our standard notation applied to the representation $(F_4, V(\pi_2))$.

Lemma 7.10. *The F_4-orbit of $[g^{-\pi_4} v^{\pi_2}]$ is open in $T\mathbb{X}$, where by $g^{-\pi_4}$ we denote a nonzero root vector of $\mathfrak{f}_4$ with weight $-\pi_4$.*

Lemma 7.11. *If all elements of $T\mathbb{X}$ have rank two or less, then the F_4-orbit of $[v^{\pi_2} + v^{-\pi_2+\alpha_2}]$ is open in $T\mathbb{X}$.*

Lemma 7.12. *The vectors $g^{-\pi_4} v^{\pi_2}$ and $v^{\pi_2} + v^{-\pi_2+\alpha_2}$ belong to different F_4-orbits.*

We present the proofs of Lemmas 7.10 and 7.11 consecutively.

Proof of Lemma 7.10. It suffices to show that the orbit $P_{\pi_2} g^{-\varepsilon_1-\varepsilon_2}$ is open in the quotient $\mathfrak{f}_4/\mathfrak{p}_{\pi_2}$, where P_{π_2} denotes the stabilizer in F_4 of v^{π_2} and $\mathfrak{p}_{\pi_2}$ is the Lie algebra of P_{π_2}. This statement follows from the fact that

$$[\mathfrak{p}_{\pi_2}, g^{-\varepsilon_1-\varepsilon_2}] + \mathfrak{p}_{\pi_2} = \mathfrak{f}_4. \qquad \square$$

Proof of Lemma 7.11. First note that, by Proposition 2.4, we have $\dim \sigma_2(\mathbb{X}) = 2\dim \mathbb{X} + 1$ and $\dim T\mathbb{X} = 2\dim \mathbb{X}$. Assume that all points in $T\mathbb{X}$ have rank two or less. This means that $T\mathbb{X}$ is contained in $\mathbb{X}_2 \sqcup \mathbb{X}$, which, by definition, is the image of $(X \times X)_0 \times \mathbb{P}^1$, where $(X \times X)_0$ is the complement of the diagonal in $X \times X$. Then the preimage of $T\mathbb{X}$ has to be an F_4-stable divisor D' of $(X \times X)_0 \times \mathbb{P}^1$. It follows from Lemma 4.1 that D' has to be a product of a divisor $D \subset (X \times X)_0$ and $\mathbb{P}^1$. Using the fact that $\pi_2 = -w_0\pi_2$ we see that there is only one F_4-stable divisor on $\mathbb{X} \times \mathbb{X} = F_4/P_{\pi_2} \times F_4/P_{\pi_2}$. This divisor has an open F_4-orbit and the image of this F_4-orbit in $\mathbb{P}(\pi_2)$ equals $F_4[v^{\pi_2} + v^{w_0 s_{\alpha_2}(\pi_2)}]$, where s_{α_2} denotes the reflection with respect to the root α_2. It remains to notice that $s_{\alpha_2}\pi_2 = \pi_2 - \alpha_2$ and that $w_0 = -1$ for F_4.

Proof of Lemma 7.12. The proof is based on the following two facts. First, the F_4-module $V(\pi_1)$ has an invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$ and thus $F_4 \subset \mathrm{SO}(V(\pi_1))$. Second, the decomposition of $\Lambda^2 V(\pi_1)$ as an F_4-module is

$$\Lambda^2 V(\pi_1) \cong V(\pi_2) \oplus V(\pi_4),$$

see [27, Table 5 on p. 305]. This allows us to represent the elements of $V(\pi_2)$ as anti-symmetric tensors and perform calculations. Essentially, we will show that

$$v^{\pi_2} + v^{-\pi_2+\alpha_2} \notin \mathrm{SO}(V(\pi_1))(g^{-\pi_4} v^{\pi_2}).$$

From now on, we consider $V(\pi_2)$ as a subspace of $\Lambda^2 V(\pi_1)$. To any $v \in \Lambda^2 V(\pi_1)$ we assign $\mathrm{Supp}\, v$ as in Sect. 6.3. It is clear that, if $v_1, v_2 \in V(\pi_2)$ are $\mathrm{SO}(V(\pi_1))$-conjugate, then the spaces $\mathrm{Supp}\, v_1$ and $\mathrm{Supp}\, v_2$ must be $\mathrm{SO}(V(\pi_1))$-conjugate and in particular $(\cdot,\cdot)$ restricted to $\mathrm{Supp}\, v_1$ and $\mathrm{Supp}\, v_2$ must have the same rank.

We have the following $\wedge$-decompositions for the vectors of $V(\pi_2) \subset \Lambda^2 V(\pi_1)$

$$v^{\pi_2} = v^{\varepsilon_1} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}, \quad v^{-\pi_2+\varepsilon_2} = v^{-\varepsilon_1} \wedge v^{\frac{-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4}{2}}.$$

We calculate

$$\begin{aligned} &v^{\pi_2} + v^{-\pi_2+\varepsilon_2} = v^{\varepsilon_1} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}} + v^{-\varepsilon_1} \wedge v^{\frac{-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4}{2}}, \\ &\mathrm{Supp}\left(v^{\varepsilon_1} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}} + v^{-\varepsilon_1} \wedge v^{\frac{-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4}{2}}\right) \\ &\quad = \left\langle v^{\varepsilon_1}, v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}, v^{-\varepsilon_1}, v^{\frac{-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4}{2}}\right\rangle \end{aligned}$$

and

$$\begin{aligned} &g^{-\pi_4} v^{\pi_2} = g^{-\varepsilon_1-\varepsilon_2}\left(v^{\varepsilon_1} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}\right) = v^{-\varepsilon_2} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}} + v^{\varepsilon_1} \wedge v^{\frac{-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}, \\ &\mathrm{Supp}\left(v^{-\varepsilon_2} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}} + v^{\varepsilon_1} \wedge v^{\frac{-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}\right) \\ &\quad = \left\langle v^{-\varepsilon_2}, v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}, v^{\varepsilon_1}, v^{\frac{-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}\right\rangle \end{aligned}$$

This implies that the space $\mathrm{Supp}(g^{-\pi_4} v^{\pi_2})$ is $(\cdot,\cdot)$-isotropic, while $\mathrm{Supp}(v^{\pi_2} + v^{-\pi_2+\varepsilon_2})$ is not. From this we conclude that $g^{-\pi_4} v^{\pi_2}$ and $v^{\pi_2} + v^{-\pi_2+\alpha_2}$ belong to different $\mathrm{SO}(V(\pi_1))$-orbits and thus to different F_4-orbits. □

7.3. R-discontinuity of $V(\pi_1)$ for E_7

This subsection is devoted to the proof of the following proposition.

Proposition 7.13. *The representation $V(\underline{\pi_1})$ of E_7 is 2-discontinuous.*

The specific notation used in this section is motivated by the guest appearance of the adjoint module of E_8 in the proof. We set $G = E_8$ and $\underline{G} = E_7$. By $\mathfrak{g}, \mathfrak{h}, \pi_1, \ldots$ and so on we denote attributes of E_8 and by $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}, \underline{\pi_1}, \ldots$ we denote the corresponding attributes of $\underline{G} = E_7$. We refer to [27, Table 1 on p. 293–295] for the roots and fundamental weights of E_8. Note that the fundamental weight π_1 of E_8 coincides with the highest root, i.e. $V(\pi_1) \cong \mathfrak{e}_8$ is the adjoint representation of E_8. The idea of the proof of Proposition 7.13 is to identify $V(\underline{\pi_1})$ of E_7 with some subspace of $\mathfrak{e}_8$ and then prove the following two lemmas.

Lemma 7.14. *a) For any $x \in X(\underline{\pi_1})$ we have* $\dim E_8x = 58$.
b) For any $x \in X_2(\underline{\pi_1})$ we have $\dim E_8x \in \{58, 92, 114\}$.

Lemma 7.15. *There exists $x \in V(\underline{\pi_1})$ such that* $\dim E_8x = 112$.

It is known that $\sigma_2(X(\underline{\pi_1})) = V(\underline{\pi_1})$ [32, Ch. III, Thm. 1.4]. Therefore $V(\underline{\pi_1})$ is r-discontinuous if and only if there exists $x \in V(\underline{\pi_1})$ such that

$$x \notin X_2(\underline{\pi_1}) \cup X(\underline{\pi_1}) \cup 0.$$

According to Lemma 7.14 and Lemma 7.15 such elements $x \in V(\underline{\pi_1})$ exist and hence Lemmas 7.14 and 7.15 imply Proposition 7.13. We now present some explanation of Lemma 7.14 and Lemma 7.15 and then proceed with their proofs.

We note the amazing fact that the $V(\underline{\pi_1})$ has only finitely many $\underline{G}$-orbits and we wish to say some words about it (see e.g. [30]). A description of the E_7-orbits on $\mathbb{F}^{56}(\dim V(\underline{\pi_1}) = 56)$ appears in [17]. The idea of the description used here comes from [2] and is related to the description of $\mathfrak{sl}_2$-triples in exceptional groups due to [11]. There is a recently developed software, which allows, in principle, to solve such problems [15].

In our proof of 2-discontinuity of $V(\underline{\pi_1})$ we use the fact that $V(\underline{\pi_1})$ is the 1-grading component of some grading of $\mathfrak{e}_8$ (representations which arise in such a way are called θ-representations, see [18,30]). We need more notation related to θ-representations.

For any $t \in \mathfrak{h}^*$ we denote by $\mathfrak{g}_t \subset \mathfrak{g}$ the corresponding weight space (we note that $\mathfrak{g}_t \neq 0$ if and only if $t \in \Delta \cup 0$). We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Cartan-Killing form and thus consider fundamental weights π_i as elements of $\mathfrak{h}^*$. We set

$$\Delta_i := \{\alpha \in \Delta \cup 0 \mid (\alpha, \pi_1) = i\}, \quad \mathfrak{g}_i := \bigoplus_{t \in \Delta_i} \mathfrak{g}_t \quad (i \in \mathbb{F}).$$

The spaces $\{\mathfrak{g}_i\}_{i \in \mathbb{F}}$ form a grading of $\mathfrak{g}$. The space $\mathfrak{g}_0$ is a Lie algebra and it acts in a natural way on $\mathfrak{g}_i$ for any $i \in \mathbb{F}$. By definition, a *θ-representation* is the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$.

We have

$$\mathfrak{g}_i = 0 \text{ if } i \notin \{-2, -1, 0, 1, 2\}, \quad \mathfrak{g}_0 \cong \mathfrak{e}_7 \oplus \mathbb{F},$$
$$\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = 1, \quad \dim \mathfrak{g}_1 = \mathfrak{g}_{-1} = 56, \quad \dim \mathfrak{g}_0 = 134.$$

We identify $\underline{\mathfrak{g}} = \mathfrak{e}_7$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$. As $\mathfrak{e}_7$-modules both $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic to $V(\underline{\pi_1})$. Further, we identify $V(\underline{\pi_1})$ with $\mathfrak{g}_1$.

The following lemma plays a key role in the proof of Lemma 7.14.

Lemma 7.16. *Let α_1, α_2 be roots of E_8 such that $\alpha_1 \neq -\alpha_2$. Then $v^{\alpha_1} + v^{\alpha_2}$ is a nilpotent element and* $\dim E_8(v^{\alpha_1} + v^{\alpha_2}) \in \{58, 92, 114\}$.

Proof. If $\alpha_1 = \alpha_2$, then $v^{\alpha_1} + v^{\alpha_2}$ is conjugate to v^{α_1}. The nilpotent element v^{α_1} is a generic nilpotent element of the corresponding Levi subalgebra with semisimple part isomorphic to A_1. Therefore $\dim E_8(v^{\alpha_1} + v^{\alpha_2}) = 58$.

If $\alpha_1 \neq \pm\alpha_2$, the vector $v^{\alpha_1} + v^{\alpha_2}$ is a nilpotent element of the Lie algebra $\mathfrak{l}_{\alpha_1,\alpha_2}$ corresponding to the root system generated by α_1, α_2. We have three possibilities: $(\alpha_1, \alpha_2) = 1$, $(\alpha_1, \alpha_2) = 0$, $(\alpha_1, \alpha_2) = -1$. In the first and third cases, we have $\mathfrak{l}_{\alpha_1,\alpha_2} \cong \mathfrak{sl}_3 = A_2$. In the second case, we have $\mathfrak{l}_{\alpha_1,\alpha_2} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = 2A_1$. For any of these Lie algebras of rank 2 it is easy to check that:

(1) if $(\alpha_1, \alpha_2) = 1$, then $v^{\alpha_1} + v^{\alpha_2}$ is conjugate in $\mathfrak{l}_{\alpha_1,\alpha_2}$ to v^{α_1} (and therefore to v^{α_2}), and thus is a distinguished nilpotent element for some root subalgebra A_1,
(2) if $(\alpha_1, \alpha_2) = 0$, then $v^{\alpha_1} + v^{\alpha_2}$ is a distinguished nilpotent element of $\mathfrak{l}_{\alpha_1,\alpha_2} \cong 2A_1$,
(3) if $(\alpha_1, \alpha_2) = -1$, then $v^{\alpha_1} + v^{\alpha_2}$ is a distinguished nilpotent element of $\mathfrak{l}_{\alpha_1,\alpha_2} \cong A_2$.

Hence $\dim E_8(v^{\alpha_1} + v^{\alpha_2}) = 58, 92, 114$, respectively, for cases 1, 2, 3, see [9, 8.4, Table: nilpotent elements for E_8]. □

Proof of Lemma 7.14. For any weight $\alpha \in \Delta_1$ and any $v^\alpha \in \mathfrak{g}_\alpha$ we have

$$v^\alpha \in \mathrm{V}(\underline{\pi_1})$$

and v^α is a highest weight vector with respect to some Borel subalgebra of $\underline{\mathfrak{g}}$, i.e.

$$v^\alpha \in X(\underline{\pi_1}).$$

On the other hand $v^\alpha \in X(\pi_1)$ and thus

$$\dim E_8 v^\alpha = \dim X(\pi_1) = 58.$$

This completes part (a).

We proceed to part (b). By Lemma 4.1, any element of $X_2(\underline{\pi_1})$ is $\underline{G}$-conjugate to the sum of two weight vectors. In our case this means that any $x \in X_2(\underline{\pi_1})$ is $\underline{G}$-conjugate to

$$v^{\alpha_1} + v^{\alpha_2}$$

for some $\alpha_1, \alpha_2 \in \Delta_1$. From this statement and Lemma 7.16 part (b) of Lemma 7.14 follows immediately. □

Proof of Lemma 7.15. We shall construct an element x with $\dim E_8 x = 112$. First note that all roots of E_8 are conjugate and that the Dynkin diagram of E_8 has a unique subdiagram of type D_4. Hence there exists roots $\alpha_1, \alpha_2, \alpha_3$ such that the quadruple

$$(-\pi_1, \alpha_1, \alpha_2, \alpha_3)$$

is a system of simple roots of Dynkin type D_4, i.e.

(1) $(-\pi_1, \alpha_i) = -1$ for $i = 1, 2, 3$,
(2) $(\alpha_i, \alpha_j) = 0$ for $i, j \in \{1, 2, 3\}, i \neq j$.

Condition 1) means that $\alpha_1, \alpha_2, \alpha_3 \in \Delta_1$. The element $v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}$ is a distinguished element of $3A_1$, where by $3A_1$ we denote the subgroup of G corresponding to the root subsystem

$$\cup_i \{-\alpha_i, \alpha_i\} \subset \Delta.$$

Therefore $\dim E_8(v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}) = 112$, see [9, 8.4, Table: nilpotent elements for E_8]. Hence, for $x = (v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}) \in V(\underline{\pi_1})$, we have $\dim E_8 x = 112$. This completes the proof. □

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